Lectures notes

on

Mechanical Vibration

By

Dr. Mihir Kumar Sutar

Asst. Professor

Mechanical Engineering Department

VSSUT Burla

VEER SURENDRA SAI UNIVERSITY OF TECHNOLOGY BURLA

LESSON PLAN

Semester: 8th Semester **Sub: Mech. Vibration**

Session: Even **Theory/Sessional: Theory**

Branch/Course: B.Tech. Mechanical Engineering

Period	Module No	Topic Name
1	I	**Damped System with Single degree of freedom:** Equilibrium method, problems
2		Viscous damping : Law of damping, problems
3		Logarithmic decrement , Steady state solution with viscous damping, problems
4		Reciprocating and Rotating unbalance, problems
5		Base excitation and Vibration Isolation, problems
6		Energy dissipated by damping, problems
7		Equivalent viscous damping. Sharpness of resonance, problems
8		Problems on viscous damping
9		Vibration measuring instruments, problems
10		Problems on vibration measuring instruments
11		Whirling of rotating shafts, problems
12		Problems on whirling of rotating shafts
13		Rigid shaft supported by flexible bearings, problems
14		Problems on rigid shaft supported by flexible bearings
Assignment 1		
15	II	**Two degree of freedom system:** Generalized derivation of equation of motion, problems.
16		co-ordinate coupling, problems
17		Langrange's equation, derivation, problems
18		Problems on Langrange's equation
19		Dynamic vibration absorber, problems
20		Application of dynamic vibration absorber, problems
21		Problems on application of dynamic vibration absorber
Assignment 2		
22	III	**Multi-degree of system:** Derivation of equation, examples of multi degrees of freedom systems, problems
23		Calculation of natural frequencies, problems
24		Problems on calculation of natural frequencies
25		Maxwell's reciprocal theorem, problems
26		Rayleigh method, problems
27		Stodala method, problems
28		matrix iteration & Holzer methods, problems
29		Problems on matrix iteration & Holzer methods
Assignment 3		

30	IV	**Torsional Vibration:** Single & multi rotor system, problems
31		Problems on single rotor systems

32		Problems on multi rotor system
33		Geared system and Branched system, problems
34		Problems on geared system
35		Problems on branched system
36		**Vibration of continuous system:** Euler equation for beam, problems
37		Problems on Euler equation for beam
38		Problems on Euler equation for beam
39		Transverse vibration of beams with different end conditions. Transverse vibration of cantilever beam, problems
40		Transverse vibration of simple supported beam and fixed beam, problems
Assignment 4		
Discussion and doubt clearing class		

MODULE 1

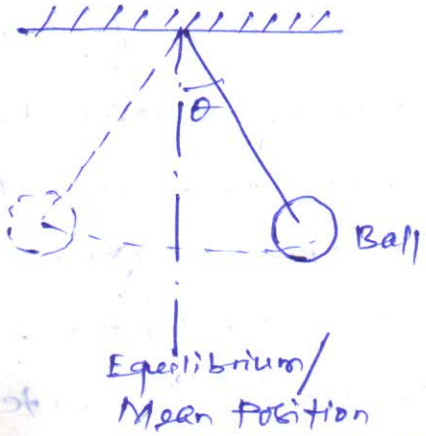
SINGLE DEGREE OF FREEDOM VIBRATION SYSTEM

MODULE-1

Basic concept of vibration / What is Vibration?

When body particles are displaced by the application of external force, the internal forces in the form of elastic energy are present in the body. These forces try to bring the body to its original position. At equilibrium position, the entire elastic energy is converted into kinetic energy and the body continues to move in the opposite direction and the process repeats.

- So any motion which repeats itself after an interval of time is called vibration; e.g. simple pendulum (shown in fig-1)



(fig-1 A simple pendulum)

Reasons of Vibrations:-

1. Unbalanced forces in the machine :- forces produced within the machine
2. Dry friction between two mating surfaces:- This produces a self excited vibration
3. External excitations:- The excitations may be periodic, random etc.
4. Earthquake: Responsible for failure of buildings/ dams etc.
5. Wind:- It may cause vibration of transmission and telephone lines under certain condition.

Definitions:-

1. Periodic motion → A motion which repeats itself after equal interval of time.

2. Time period → Time taken to complete one cycle.

3. Frequency → No of cycles/unit time

4. Simple Harmonic Motion → A periodic motion of a particle whose acceleration is always directed towards the mean position.

5. Amplitude of motion → Maximum displacement of a vibrating body from mean position

6. free vibrations → Vibration of a system because of its own elastic property without any external exciting forces acting on it.

7. Forced vibration → The vibrations the system executes under the action of an external periodic force. The frequency of vibration is same to that of excitation.

8. Natural frequency → Frequency of free vibration of the system. It is constant for a given system.

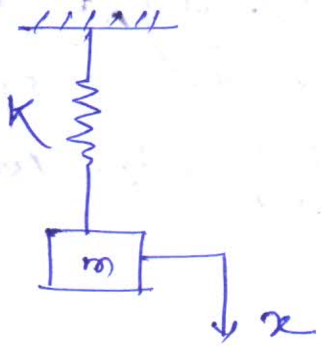
9. Resonance → Vibration of a system ~~when~~ in which the frequency of external force is equal to the natural frequency of the system.

10. Damping → Resistance to the motion of the vibrating body.

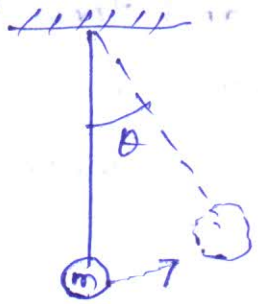
11. Degree of freedom → No of independent coordinates required to specify completely the configuration of the system at any instant.

Few examples of Single degree of freedom system, have been

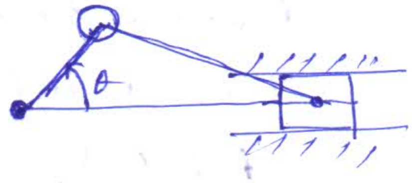
Example :-



(a) Spring mass system



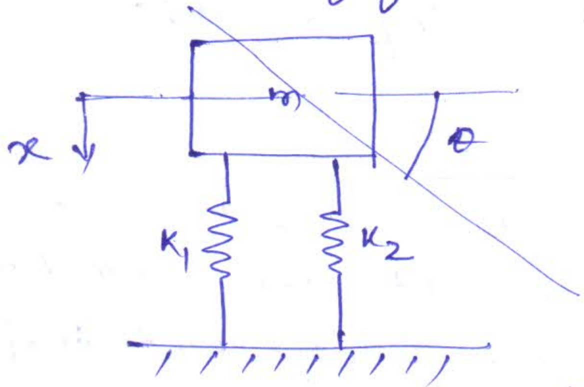
(b) Simple pendulum



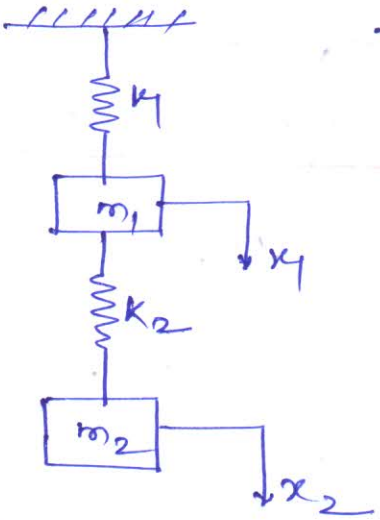
(c) Crank slider Mechanism.

(Fig 2 :- Examples of Single Degrees of freedom system.)

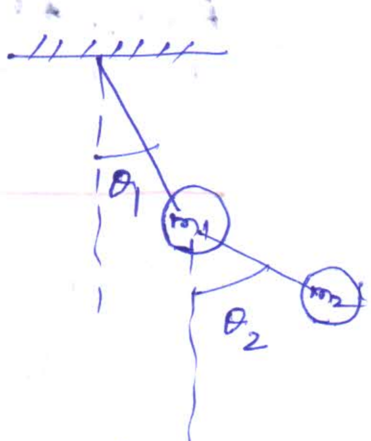
shown in fig. (2). And fig (3) depicts few examples of two degrees of freedom system.



(a) Spring supported Rigid mass.

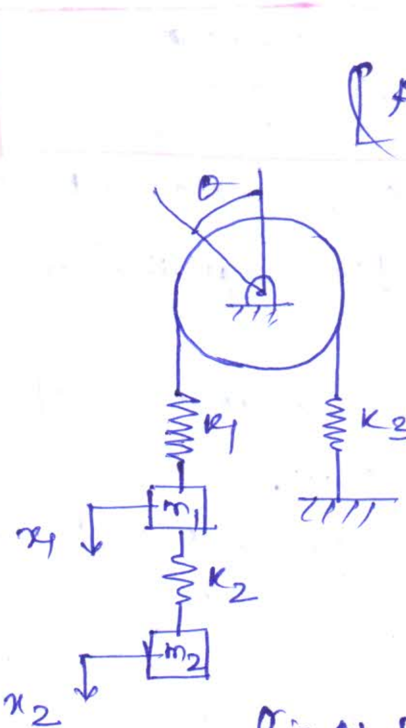


(b) Two mass two spring system.



(c) Double pendulum system.

[Fig - 3 :- Examples of two degrees of freedom (DOF) systems]



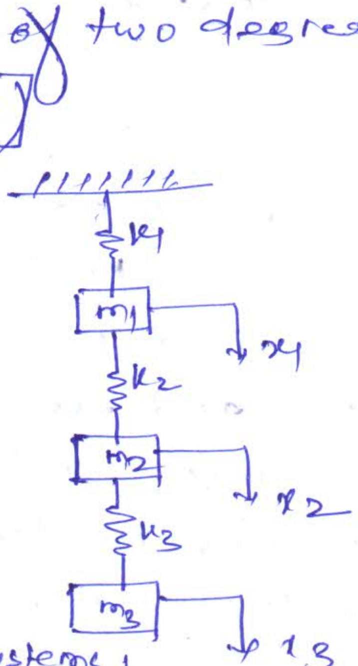
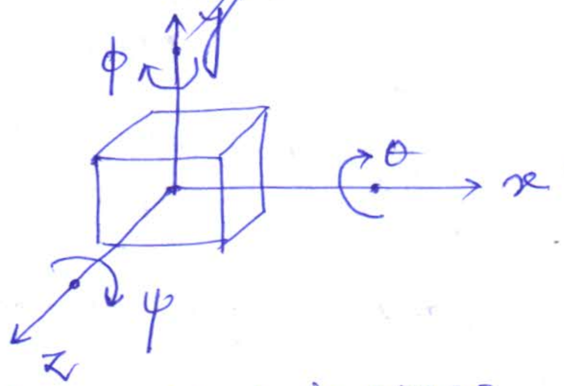


Fig 4: Example of 3 DOF systems.

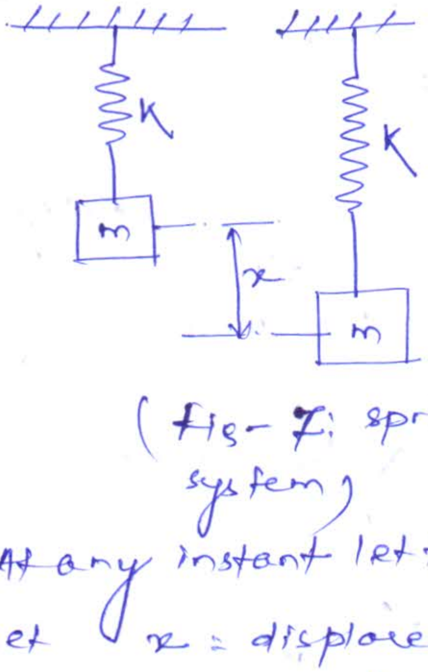
Similarly a ~~rigid~~ body in space has six dof (i.e. three translational and three rotational), as shown fig. 5. And a flexible beam both two supports has a infinite no of degrees of freedom (shown in fig. 6).



(Fig 5:- A body in space having 6-dof)

(Fig. 6: A flexible beam having ~~six dof~~ infinite dofs)

Derivation of Differential Equation:-



(Fig- 7: spring mass system)

Consider a spring mass system (fig 7) constrained to move in a rectilinear manner along the longitudinal axis.

Let m = mass of the block attached to spring

k = Spring stiffness.

Sign convection →   Downward = +ve   Upword = −ve

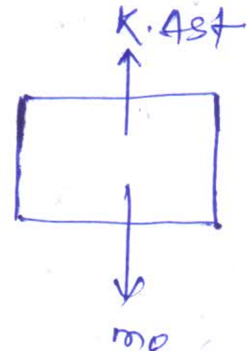
- At any instant let the mass occupy any displaced position,

Let x = displacement of mass m from equilibrium position.

Considering displacement x to be +ve in downward direction and −ve in the upward direction.

For an initial infinitesimal displacement of Δst, prior to x displacement, in the equilibrium position the forces acting on the mass are:

(i) mg → vertically downward
(ii) $K.\Delta st$ → spring force, vertically upward.



(fig 8: Eq. position of mass)

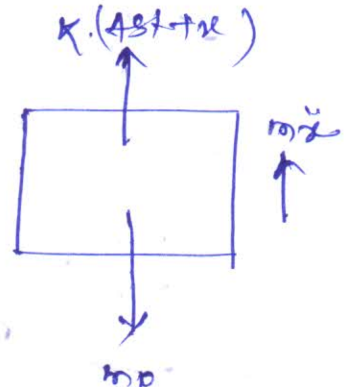
For equilibrium

$$mg = K\Delta st \quad \text{— (1)}$$

And after a displacement of 'x'

total spring force $= K.(\Delta st + x)$

And the forces acting on the mass from Newton's 2^{nd} law of motion



(fig. 9: forces after x displacement)

$$m\ddot{x} = mg - K.(\Delta st + x)$$
$$= mg - K.\Delta st - Kx$$
$$= \cancel{mg} - \cancel{mg} - Kx \quad \{\text{from eq. (1)}\}$$

$$\Rightarrow \boxed{m\ddot{x} + Kx = 0} \quad \text{— (2)}$$

Solution of Differential Equation:-

We have the differential equation for the spring mass system

$$m\ddot{x} + Kx = 0$$

It is an equation of simple harmonic motion.

~~The solution of the above equation will be~~

$$\cancel{x = A\cos\omega_n t + B\sin\omega_n t \quad \text{— (2)}}$$

Now from eq (1) we have

$$\ddot{x} + \frac{K}{m}x = 0 \quad \text{— (3)}$$

let $\frac{K}{m} = \omega_n^2$

so equation (3) may be written as:

$$\boxed{\ddot{x} + \omega_n^2 x = 0} \quad \text{— (4)}$$

~~Eq. (4) has a solution as in eq. (2)~~

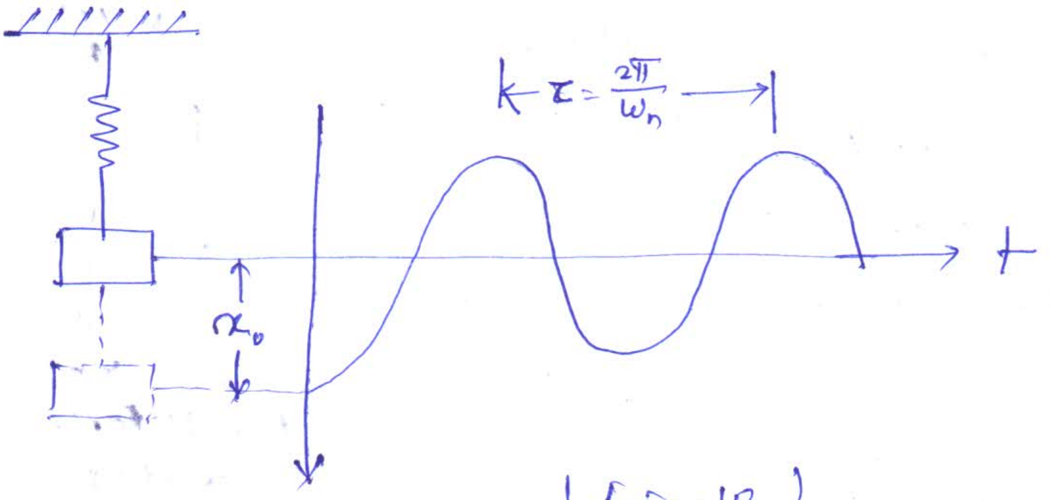
$$\cancel{x = A\cos\omega_n t + B\sin\omega_n t}$$

The standard solution for this differential equation is written as

$$x = A \sin \omega_n t + B \cos \omega_n t \quad \text{--- (5)}$$

Where A and B are constants, whose value can be obtained from initial conditions.

$$\left.\begin{array}{ll} x = x_0 , & \text{at } t = 0 \\ \dot{x} = 0 , & \text{at } t = 0 \end{array}\right\} \text{--- (6)}$$



(fig. 10)

Differentiating equation (5)

$$\dot{x} = A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t \quad \text{--- (7)}$$

substituting the initial condition in eq. (5) and eq. (7)

$$X_0 = 0 + B$$

$$0 = A\omega_n - 0$$

Gives $\boxed{\begin{array}{l} A = 0 \\ B = X_0 \end{array}}$ ——— (8)

Substituting the values of these constants, we have

$$\boxed{x = X_0 \cos \omega_n t} \quad \text{——— (9)}$$

Equation (9) is the final solution for the specified initial condition.

The time period for one complete cycle of 2π rad is

$$\tau = \frac{2\pi}{\omega_n}$$

Natural frequency is the inverse of time period

$$f_n = \frac{\omega_n}{2\pi}$$

$$\Rightarrow \boxed{\omega_n = \sqrt{\frac{K}{m}}}$$

Therefore $f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = \frac{1}{2\pi}\sqrt{\frac{K \cdot g}{m \cdot g}} = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta st}}$ $\left(\because \Delta st = \frac{mg}{K}\right)$

or $\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{9.8}{\Delta st}} = \frac{0.4892}{\sqrt{\Delta st}}}$ Hz. — (10)

<u>Example-1</u> A light cantilever of length l has a mass M fixed at it's free end. Find the frequency of lateral vibration in the vertical plane.

The deflection at the free end of the cantilever

$$\Delta st = \frac{M \cdot g l^3}{3EI} \quad \text{— (1)}$$

Where E = modulus of elasticity

I = MI of the section of beam about it's neutral axis.

Now stiffness $K = \frac{Mg}{\Delta st} = Mg \times \frac{3EI}{Mgl^3} = \boxed{\frac{3EI}{l^3}}$

and circular frequency $\omega_n = \sqrt{\frac{K}{m}}$

$$= \sqrt{\frac{3EI/l^3}{M}} = \sqrt{\frac{g}{\Delta st}}$$

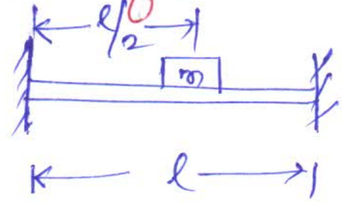
$$\Rightarrow \cancel{\boxed{\omega_n = \sqrt{\frac{g}{\Delta st}} \text{ rad/sec}}}$$

$$\Rightarrow \boxed{\omega_n = \sqrt{\frac{3EI}{l^3 M}}}$$

And natural frequency $\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{3EI}{Ml^3}}}$ Hz.

Example-2

Find the natural frequency of the sys. shown in the figure.



Deflection at the centre of a beam fixed at both ends and a central load W is

$$\Delta st = \frac{Wl^3}{192EI}$$

and stiffness $K = \frac{\text{load}}{\text{deflection}} = \frac{W}{Wl^3/192EI}$

$$\Rightarrow \boxed{K = \frac{192EI}{l^3}}$$

General equation for undamped free vibration is

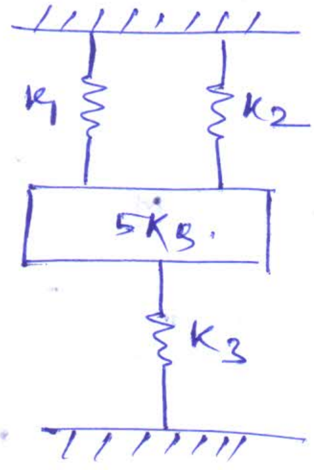
$$m\ddot{x} + Kx = 0$$

and $\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{192EI}{ml^3}}$ rad/sec.

So Natural frequency $f_n = \frac{1}{2\pi}\sqrt{\frac{192EI}{ml^3}}$ Hz.

Example-3

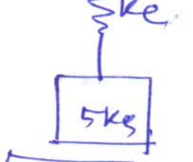
Find the natural frequency of the system shown in figure. Given $k_1 = k_2 = 1500$ N/m, $K_3 = 2000$ N/m, $m = 5$ kg.



The equivalent stiffness for springs in parallel

$K_e = K_1 + K_2 + K_3 = 1500 + 1500 + 2000$

$= 5000$ N/m.

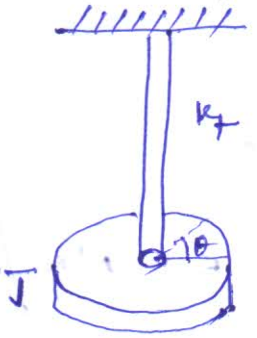


So $\omega_n = \sqrt{\frac{Ke}{m}} = \sqrt{\frac{5000}{5}} = 31.62$ rad/sec.

$f_n = \frac{1}{2\pi}\sqrt{1000} = 5.03$ Hz. (Ans).

Torsional vibrations:-

Considering a rotor ~~mass~~ having a mass moment of inertia J, connected to a shaft of torsional stiffness K_t (as shown in fig-11). When the rotor is displaced in an angular manner, it executes torsional vibrations.
- It's natural frequency can be obtained in the following manner:



(fig-11 Torsional system)

At any instance the rotor occupies a position θ with reference to the equilibrium position.
The torque acting on the rotor $= -K_t\theta$

-ve sign indicates the torque acts on the rotor in opposite direction to that of the twist.

$$J\ddot{\theta} = -K_t\theta \quad \text{---} (1)$$

or $$J\ddot{\theta} + K_t \cdot \theta = 0$$

or $$\ddot{\theta} + \left(\frac{K_t}{J}\right)\theta = 0 \quad \text{---} (2)$$

substituting $\omega_n^2 = K_t / J \quad \text{---} (3)$

So equation (2) becomes

$$\boxed{\ddot{\theta} + \omega_n^2\theta = 0} \quad \text{---} (4)$$

Natural frequency of vibration of this system can be obtained from the equation

$$\boxed{\omega_n = \sqrt{K_t/J}} \quad \text{---} (5)$$

so $$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{K_t}{J}}}$$

Example-4 Calculate the natural frequency of vibration of a torsional pendulum with following dimensions:

length of the rod, $l = 1$ m.
Diameter of rod, $d = 5$ mm
Diameter of rotor, $D = 0.2$ m
Mass of rotor, $M = 2$ kg.

The modulus of rigidity for the material of rod may be assumed to be $0.83 \times 10^{11}\ N/m^2$

Solu^n We have mass moment of inertia $J = \frac{1}{2} mr^2$

$\Rightarrow J = \frac{1}{2} M \left(\frac{D}{2}\right)^2 = \frac{1}{2} \times 2 \times (0.1)^2 = 0.01\ kg\,m^2$

Now using the relation

$$\frac{T}{I_p} = \frac{G\theta}{l}$$

or, torsional stiffness $k_t = \frac{T}{\theta} = \frac{G \cdot I_p}{l}$

so $k_t = \dfrac{0.83 \times 10^{11} \times \frac{\pi}{32} \times (0.005)^4}{1} = 5.09\ Nm/rad$

so $\omega_n = \sqrt{\dfrac{k_t}{J}} = \sqrt{\dfrac{5.09}{0.01}} = 22.6\ rad/sec$

$f_n = 22.6/2\pi = \boxed{3.59\ Hz}$

Energy Method:-

free vibration of systems involves the cyclic interchange of KE and PE. In undamped free vibrating systems no energy is dissipated or removed from the system. The KE, T is stored in the mass by virtue of it's velocity and potential energy U is stored in the form of strain energy in elastic deformation. As the total energy in the system is constant the principal of conservation of mechanical energy applies. Since the mechanical energy is conserved, the sum of KE and PE is constant and it's rate of change of is zero.

The principle can be expressed as

$$\boxed{\begin{array}{l} T + U = \text{constant} \\ \dfrac{d}{dt}(T+U) = 0 \end{array}}$$

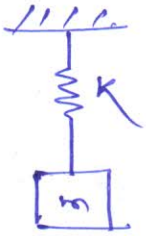
Energy Method :-

free vibration of systems involves the cyclic interchange of KE and PE. In undamped free vibrating systems, no energy is dissipated or removed from the system. The KE ~~is stored~~ T is stored in the mass by virtue of it's velocity and potential energy U is stored in the form of strain energy in elastic deformation. As the total energy in the system is constant, the principle of conservation of mechanical energy applies. Since the mechanical energy is conserved, the sum of KE and PE is const. and it's rate of change is zero.

The principle can be expressed as:

$$T+U = \text{constant}$$

$$\frac{d}{dt}(T+U) = 0$$



For the system shown in the figure

$$T = \frac{1}{2} m\dot{x}^2$$

$$U = \frac{1}{2} Kx^2$$

$$\frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} Kx^2\right) = 0$$

$$\Rightarrow (m\ddot{x}\dot{x} + K\dot{x}x) = 0$$

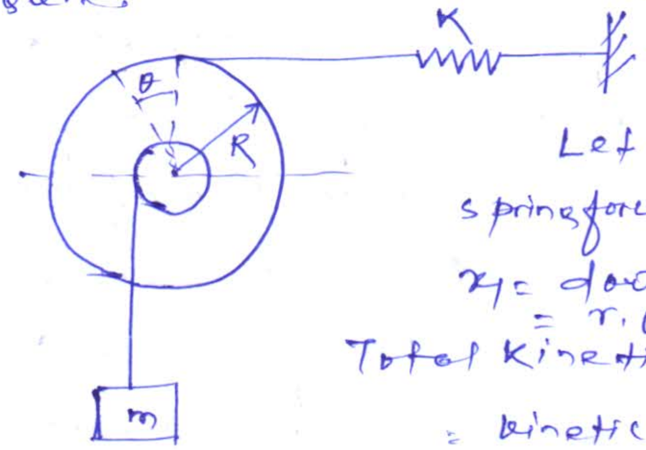
$$\Rightarrow \boxed{m\ddot{x} + Kx = 0}$$

And $\omega_n = \sqrt{\frac{K}{m}}$

natural frequency $\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}}$

Example - 1

Find the natural frequency of the system shown in the figure.



Let x_2 = spring deflection

spring force $K x_2 = KR\theta$

x_1 = downward movement of mass m $= r\theta$

Total Kinetic energy

= kinetic energy of mass + kinetic energy of rotating element

$= \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} I \dot{\theta}^2$

Potential energy of spring $= \frac{1}{2} K x_2^2$

Total energy $= \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} K x_2^2$

By energy method, we have

$$\frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} K x_2^2 = \text{constant} \quad \text{--- (1)}$$

~~Differenciating eq. (1) wrt time~~

It may be represented as

$$\frac{1}{2} mr^2\dot{\theta}^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} KR^2\theta^2 = \text{constant}$$

$$\frac{d}{d\theta}\left[\frac{1}{2} mr^2\dot{\theta}^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} KR^2\theta^2\right] = \frac{d}{d\theta}(\text{constant})$$

$$mr^2\dot{\theta}\ddot{\theta} + I\dot{\theta}\ddot{\theta} + KR^2\theta\dot{\theta} = 0$$

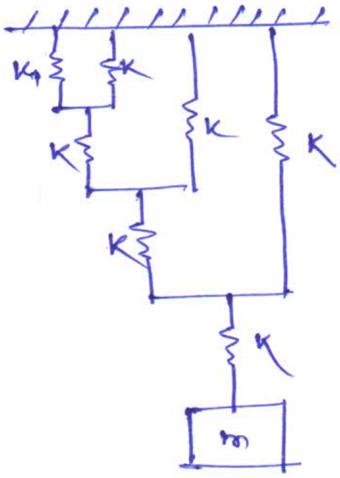
$$(mr^2 + I)\ddot{\theta} + KR^2\theta = 0$$

or $$\ddot{\theta} + \left(\frac{KR^2}{mr^2+I}\right)\theta = 0$$

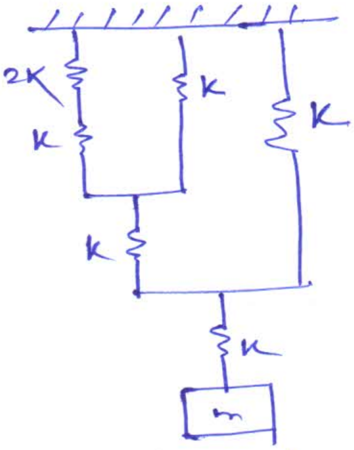
$$\omega_n = \sqrt{\left(\frac{KR^2}{mr^2+I}\right)} \quad \text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{KR^2}{mr^2+I}}$$

(Ans)

Example - 2 find the natural frequency of the system shown in the figure. Take $K = 2\times10^{5}\ N/m$, $m = 20\ kg$.



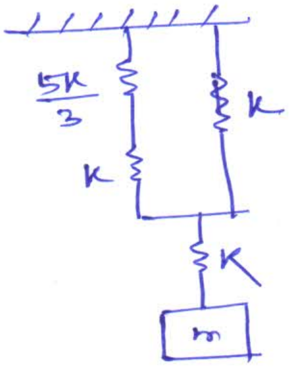
$\simeq$



$$\frac{1}{K_{e1}} = \frac{1}{2K} + \frac{1}{K} \Rightarrow \frac{1}{K_{e1}} = \cancel{3K}\left(\frac{1+2K}{2K}\right) \Rightarrow K_{e1} = \frac{2K}{\cancel{(1+2K)}\,3}$$

$$\cancel{K_{e2} = \frac{2K}{1+2K} + K = \frac{2K+K+2K^2}{1+2K}}$$

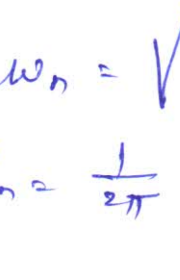
$$K_{e2} = \frac{2K}{3} + K = \frac{5K}{3}$$



$$\frac{1}{K_{e3}} = \frac{3}{5K} + \frac{1}{K} = \frac{3+5}{5K}$$

$$K_{e3} = \frac{5K}{8}$$

$$K_{e4} = \frac{5K}{8} + K = \frac{13K}{8}$$



$$\frac{1}{K_{e5}} = \frac{8}{13K} + \frac{1}{K} = \frac{8+13}{13K}$$

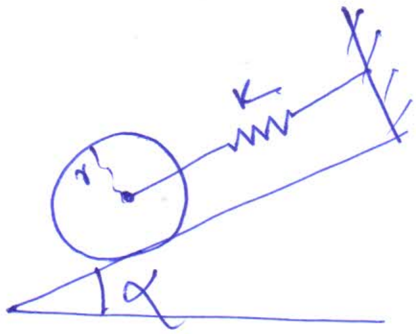
$$K_{e5} = \frac{13K}{21}$$

$$\omega_n = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{13\times2\times10^5}{20\times21}} = 78.68\ \text{rad/sec}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_e}{m}} = 12.5\ \text{Hz}.$$

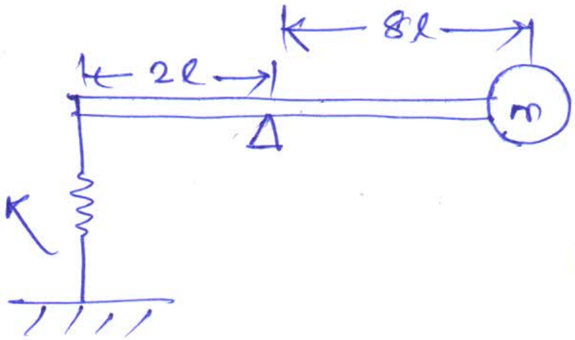
Assignment - 1

(1)



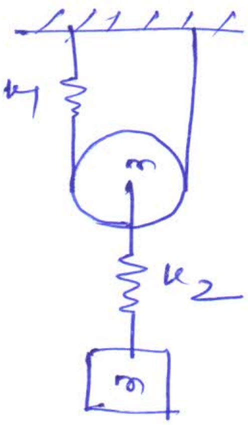
A circular cylinder of radius r and mass m is connected by a spring of stiffness K on an inclined plane. If it is free to roll on the rough surface which is without slipping, determine the natural frequency.

(2)



Find the natural frequency of the system if m = 10 kg, K = 1000 N/m

(3)



Determine the natural frequency of the mass m = 15 kg,

$k_1 = 8 \times 10^3$ N/m

$k_2 = 6 \times 10^3$ N/m,

FREE DAMPED VIBRATION

In many practical systems, the vibrational energy is gradually converted to heat or sound. Due to the reduction in the energy, the response, such as the displacement of the system, gradually decreases. The mechanism by which the vibrational energy is gradually converted into heat or sound is known as damping. Although the amount of energy converted into heat or sound is relatively small, the consideration of damping becomes important for an accurate prediction of the vibration response of a system. A damper is assumed to have neither mass nor elasticity, and damping force exists only if there is relative velocity between the two ends of the damper. It is difficult to determine the causes of damping in practical systems. Hence damping is modeled as one or more of the following types.

Types of Damping

1. Viscous damping
2. Coulomb damping
3. Structural damping
4. Slip or interfacial damping

1. Viscous damping

Viscous damping is the most commonly used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water, or oil, the resistance offered by the fluid to the moving body causes energy to be dissipated. In this case, the amount of dissipated energy depends on many factors, such as the size and shape of the vibrating body, the viscosity of the fluid, the frequency of vibration, and the velocity of the vibrating body. In viscous damping, the damping force is proportional to the velocity of the vibrating body. Typical examples of viscous damping include (1) fluid film between sliding surfaces, (2) fluid flow around a piston in a cylinder, (3) fluid flow through an orifice, and (4) fluid film around a journal in a bearing.

2. Coulomb damping

Here the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused by friction between rubbing surfaces that either are dry or have insufficient lubrication.

3. Structural damping

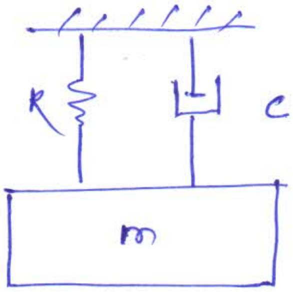
When a material is deformed, energy is absorbed and dissipated by the material. The effect is due to friction between the internal planes, which slip or slide as the deformations take place.

When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop. The area of this loop denotes the energy lost per unit volume of the body per cycle due to damping

4. Slip or interfacial damping

Microscopic slip occur on the interfaces of machine elements in contact under fluctuating loads. The amount of damping depends upon the material combination, surface roughness at interface, contact pressure and the amplitude of vibration.

Differential equation of free damped Vibration



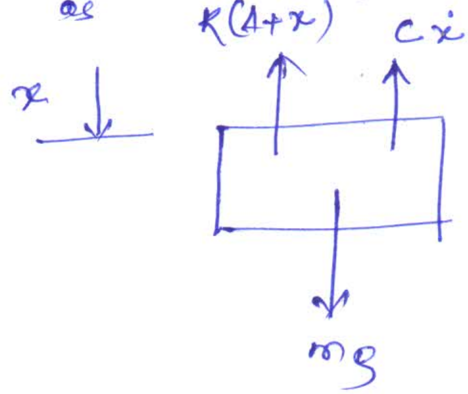
In the study of vibration, the process of energy dissipation is generally referred to as damping. The most common phenomenon of energy dissipating element is viscous damper, also called dashpot.

Viscous damping force is proportional to the velocity $\dot{x}$ of the mass and acts in the direction opposite to the velocity of the mass m. It can be expressed as,

$$F = c\dot{x} \quad \text{---}(1)$$

Where c = damping coefficient of viscous damping.

The free body diagram of the system can be represented as



Applying Newton's second law

$$m\ddot{x} = -K(\Delta + x) + mg - c\dot{x}$$

$$\Rightarrow m\ddot{x} = -\cancel{K\Delta} - Kx + \cancel{mg} - c\dot{x}$$

$$= -Kx - c\dot{x}$$

$$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + Kx = 0} \quad \text{---}(2)$$

or $$\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{K}{m}\right)x = 0 \quad \text{---}(3)$$

Eq. (3) is the differential equation of motion for free vibration of a damped spring-mass system.

Assuming a solution in the form $x(t) = Ce^{st}$ to obtain the auxiliary equation

$$s^2 + \frac{c}{m}s + \frac{K}{m} = 0 \quad \text{---}(4)$$

Eq.(4) has roots

$$s_{1,2} = \frac{1}{2}\left[-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{K}{m}}\right]$$

or

$$\boxed{s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}}} \quad \text{--- (5)}$$

The solution of eq. (5) takes one of the three forms, depending on whether the quantity $\left[\left(\frac{c}{2m}\right)^2 - \frac{K}{m}\right]$ is zero, positive or negative.

If $\left(\frac{c}{2m}\right)^2 - \frac{K}{m} = 0$ we have,

$$\frac{c}{2m} = \sqrt{\frac{K}{m}} = \omega_n$$

$$\Rightarrow \boxed{c = 2m\omega_n} \quad \text{--- (6)}$$

in which case we have repeated roots

$s_1 = s_2 = -\frac{c}{2m}$ and the solution is

$$\boxed{x(t) = (A+B)\,e^{-(c/2m)t}} \quad \text{--- (7)}$$

In this particular case, the damping constant or coefficient is called critical damping constant denoted by

$$\boxed{C_c = 2m\omega_n} \quad \text{--- (8)}$$

And eq. (5) may be written as

$$s_{1,2} = -\frac{c}{c_c}\omega_n \pm \omega_n\sqrt{\left(\frac{c}{c_c}\right)^2 - 1}$$

or

$$\boxed{s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n} \quad \text{--- (9)}$$

Where $\omega_n = \sqrt{\frac{k}{m}}$, circular frequency of the corresponding undamped system and

$$\boxed{\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n}} \quad \text{——— (10)}$$

and ζ = damping factor.

Case - I when $\zeta < 1$

If $\zeta < 1$ both the roots in eq. (9) are imaginary and given by

$$s_{1,2} = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)\omega_n \quad \text{——— (11)}$$

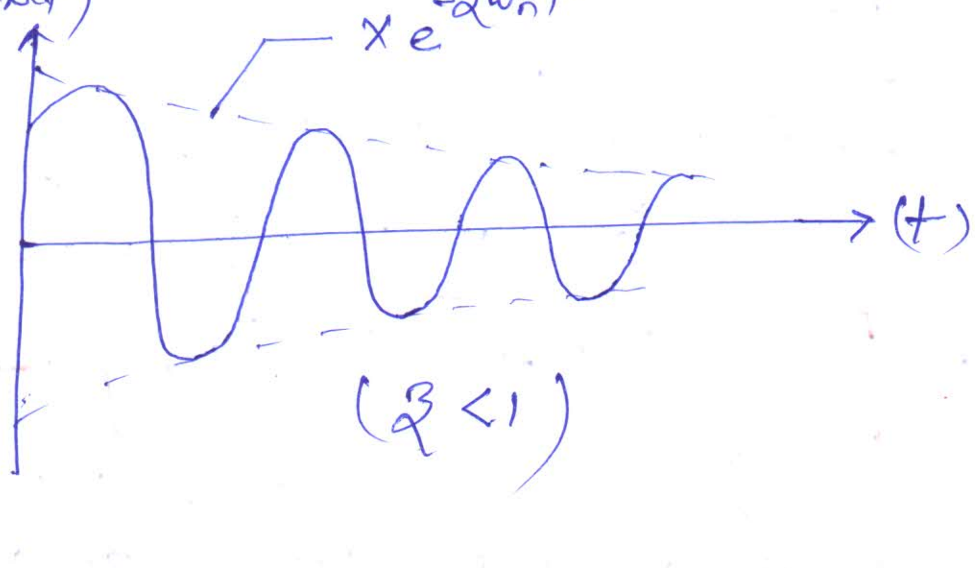
and the solution of motion is

$$\boxed{x(t) = X e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)} \quad \text{——— (12)}$$

where ω_d = damped circular frequency (which is always less than ω_n)

ϕ = phase angle of damped oscillation.

The function is a harmonic function whose amplitude decays exponentially with time. The general form of motion is shown in the figure and the system is said to be underdamped.



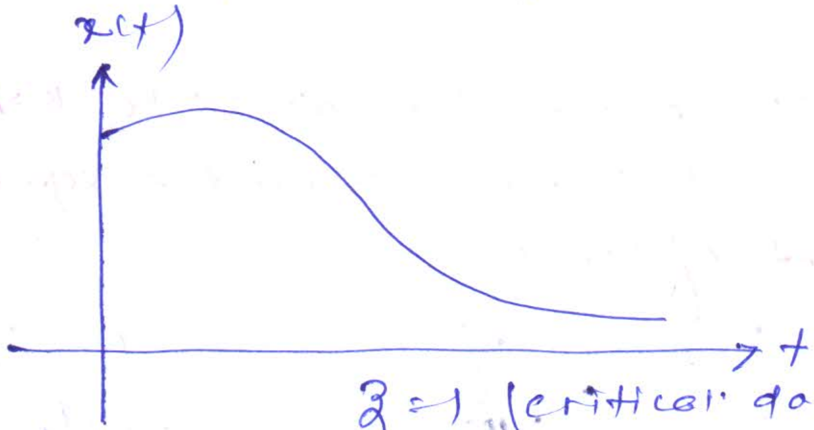
$(\zeta < 1)$

Case 2:- $\zeta = 1$ or $c = c_c = 2m\omega_n$

If $\zeta = 1$, the damping constant is equal to the critical damping constant and the system is called to be critically damped.

The displacement equation (7) may be written as

$$x(t) = (A + Bt)e^{-\omega_n t} \quad \text{—(13)}$$



$\zeta = 1$ (critical damping)

The solution to the above equation (13) is the product of a linear function of time and decaying exponential.

Case - 3 $\zeta > 1$ or $c > 2m\omega_n$

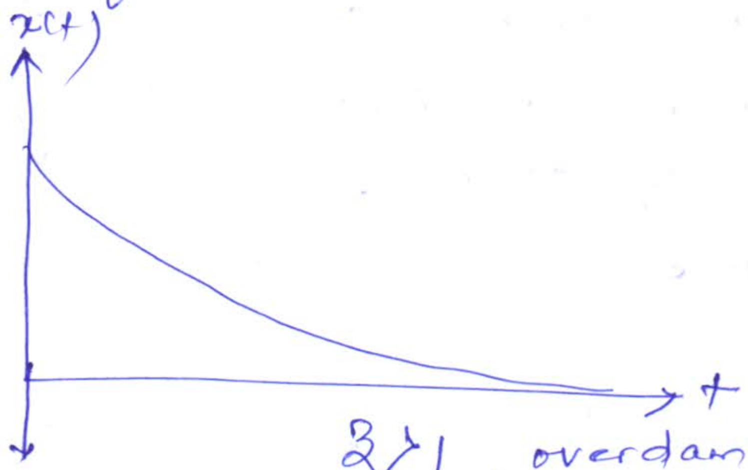
If $\zeta > 1$ the system is called overdamped. Here both the roots are real and are given by

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

Since $\sqrt{\zeta^2 - 1} < \zeta$, it can be seen that both s_1 and s_2 are negative so that the displacement is the sum of two decaying exponentials given by

$$\boxed{x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}} \quad \text{—(14)}$$

The motion will be non oscillating and shown in figure.



$\zeta > 1$, overdamping.

Example - 1

A damped spring-mass has $m = 12$ kg, $k = 12$ N/mm and $c = 0.3$ Ns/mm. Obtain the equation of displacement of the mass.

The natural frequency of the undamped system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12 \times 1000}{12}} = 31.62 \text{ rad/sec}$$

critical damping constant $c_c = 2m\omega_n$

$$= 2 \times 12 \times 31.62 = 758.95 \text{ Ns/m or}$$

$$0.759 \text{ Ns/mm},$$

And damping factor $\zeta = \frac{c}{c_c} = \frac{0.3}{0.759} = \boxed{0.395}$

As the system is underdamped ($\because \zeta < 1$)

the damped natural frequency $\omega_d = (\sqrt{1-\zeta^2})\omega_n$

$$= \{\sqrt{1-(0.395)^2}\} 31.62 = 29.05 \text{ rad/s.}$$

and $\zeta\omega_n = 0.395 \times \cancel{}\, 31.62 = 11.47$

Equation of displacement

$$x(t) = Xe^{-11.47t} \sin(29.05t + \phi)$$

$$\boxed{Xe^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}$$

Example - 2

A single dof viscously damped system has a spring stiffness of 6000 N/m, critical damping constant of 0.3 Ns/mm and a damping ratio of 0.3. If the system is given an initial velocity of 1 m/s, determine the max^m displacement of the system.

The natural frequency of the system $\omega_n = \sqrt{\frac{k}{m}}$

We have $C_c = 0.3\,Ns/mm = 300\,Ns/m = 2m\omega_n$

$$= 2m\sqrt{\frac{k}{m}} = 2\sqrt{6000m}$$

$$\Rightarrow 300 = 2\sqrt{6000m} \Rightarrow m = \boxed{3.75\,kg.}$$

$$\omega_n = \sqrt{\frac{6000}{3.75}} = 40\,rad/sec.$$

Damping ratio $\xi = \frac{\omega}{\omega_n} = 0.3$

or $C = C_c \times 0.3 = 0.3 \times 0.3 = 0.09\,Ns/mm$

$$= 900\,Ns/m;$$

Assuming $x_0 = 0$ and $\dot{x}_0 = 1\,m/s$. the general expression for displacement is

$$x(t) = e^{-\xi\omega_n t}\frac{\dot{x}_0}{\omega_n\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\omega_n t$$

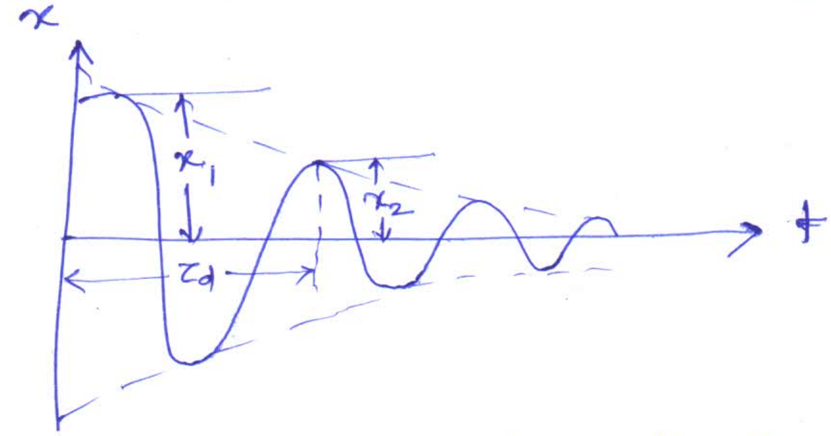
For max^m displacement (x_{max}) $\omega_n t = \pi/2$

and $\sin\sqrt{1-\xi^2}\,\omega_n t = 1$

$$x_{max} = e^{-0.3\times\left(\frac{\pi}{2}\right)}\frac{1}{40\sqrt{1-0.3^2}}(1) = \boxed{0.01636\,m}$$

Logarithmic Decrement :-

The logarithmic decrement represents the rate at which the amplitude of a free damped vibration decreases. It is defined as the ratio of any two successive amplitudes on the same side of the mean line.



In other words we can say it is defined as the natural logarithm of the ratio of any two successive amplitudes.

The displacement of an underdamped system is a sinusoidal oscillation with decaying amplitude as shown in the figure.

The ratio of successive amplitude is

$$\boxed{\frac{x_i}{x_{i+1}} = \frac{Xe^{-\xi \omega_n t_i}}{Xe^{-\xi \omega_n (t_i+\tau_d)}} = e^{\xi \omega_n \tau_d} = \text{constant}} \quad \text{---(1)}$$

So $$\boxed{\frac{x_i}{x_{i+1}} = e^{\xi \omega_n \tau_d}} \quad \text{---(2)}$$

Now substituting $\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\xi^2}}$ in eq.(2)

$$\frac{x_i}{x_{i+1}} = e^{\xi . \omega_n . \frac{2\pi}{\omega_n \sqrt{1-\xi^2}}} = e^{\left(2\pi\xi / \sqrt{1-\xi^2}\right)}$$

and for small damping

$$\boxed{\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = 2\pi\xi} \quad \text{---(3)}$$

$\because$ If ζ is small then $\delta = 2\pi\zeta$

Since $\sqrt{1-\zeta^2} \simeq 1$

From equation (3) we have

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

or $$\zeta = \frac{\delta \cdot \sqrt{1-\zeta^2}}{2\pi}$$

or $$\zeta^2 = \frac{\delta^2(1-\zeta^2)}{(2\pi)^2}$$

$$\Rightarrow (2\pi)^2 \cdot \zeta^2 = \delta^2 - \delta^2\zeta^2$$

$$\Rightarrow (2\pi)^2 \cdot \zeta^2 + \delta^2\zeta^2 = \delta^2$$

$$\Rightarrow \zeta^2\left[(2\pi)^2 + \delta^2\right] = \delta^2$$

$$\Rightarrow \boxed{\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}} \quad \text{— (4)}$$

Also $\boxed{\zeta = \frac{\delta}{2\pi}}$ (For small damping) — (5)

- Logarithmic decrement can also be calculated from the ratio of amplitudes of several cycles apart.

Thus if x_n is the amplitude n cycles after x_0, then

$$\frac{x_0}{x_n} = \frac{x_0}{x_1} \cdot \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdots \frac{x_{n-1}}{x_n}$$

$$= \left(\frac{x_j}{x_{j+1}}\right)^n$$

Natural log of the ratio $\ln\left(\frac{x_0}{x_n}\right) = n \ln\left(\frac{x_j}{x_{j+1}}\right)$

$$\Rightarrow \boxed{\ln\left(\frac{x_0}{x_n}\right) = n \cdot \delta} \quad \text{— (6)}$$

or $$\boxed{\delta = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)} \quad -(7)$$

So logarithmic decrement δ can be obtained from the amplitude loss occuring over several cycles

$$\boxed{n = \frac{1}{\delta}\ln\left(\frac{x_0}{x_n}\right) = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{x_0}{x_n}\right)} \quad -(8)$$

Equation (8) is used to determine no of cycles required for a given system to reach a specified reduction in amplitude

Example A single dof viscous damping system makes 5 complete oscillations/second. Its amplitude diminishes to 15% in 60 cycles. Determine

(a) logarithmic decrement

(b) damping ratio.

(a) Data given $f = 5$

$$\tau_d = \frac{1}{5} = 0.2 \text{ sec.}$$

But $\tau_d = \frac{2\pi}{\omega_d} \Rightarrow \omega_d = \frac{2\pi}{0.2} = 31.416 \text{ rad/s}$

Logarithmic decrement $\delta = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)$

$$\delta = \frac{1}{60}\ln(0.15) = 0.0451$$

(b) damping ratio (ζ)

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.0451}{\sqrt{(2\pi)^2 + 0.0451^2}} = 0.007177$$

(Ans)

(c

Example-2 . A single dof spring mass damper has a mass of 60 kg and a spring stiffness of 6000 N/m. Determine the following

(a) critical damping coefficient

(b) damped natural frequency when $c = 2c_c/3$

(c) logarithmic decrement.

(a) ~~Ce = 2~~ $m = 60$ kg. $K = 6000$ N/m.

$$c_c = 2m\omega_n = 2m.\sqrt{\frac{K}{m}} = 2\sqrt{Km}.$$

$$= 2\times\sqrt{6000\times 60} = 1200 \text{ Ns/m}.$$

(b) Now, $c = 2.\frac{c_c}{3} = 800$ Ns/m,

Damed natural frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{K}{m}}\sqrt{1-\left(\frac{c}{c_c}\right)^2}$$

$$= \sqrt{\frac{6000}{60}}\sqrt{1-\left(\frac{800}{1200}\right)^2}$$

$$= 7.45 \text{ rad/sec.}$$

(c) Logarithmic decrement

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi\left(\frac{2}{3}\right)}{\sqrt{2\pi-(2/3)^2}}$$

$$= \boxed{5.6198}$$

Question - 1

A damper offers resistance of 0.05N at constant velocity 0.04 m/s. The damper is used with $k = 9 N/m$. Determine the damping and frequency of the ~~mass of the~~ system if the mass is 0.1 kg.

We have Damping force $F = c\dot{x}$

$\dot{x} = 0.04 \, m/s$. $F = 0.05N$.

$$C = \frac{F}{\dot{x}} = \frac{0.05}{0.04} = 1.25 \, Ns/m.$$

$$C_c = 2\sqrt{Km} = 2 \times \sqrt{9 \times 0.1} = 1.897 \, Ns/m.$$

$$\text{damping factor } \xi = \frac{C}{C_c} = \frac{1.25}{1.897} = 0.658$$

So the system is under damped

$$\omega_d = \omega_n \sqrt{1-\xi^2} = \sqrt{\frac{K}{m}} \sqrt{1-0.658^2}$$

$$= \sqrt{\frac{9}{0.1}} \sqrt{1-0.658^2}$$

~~Q.2~~ A vibrating system is defined by the following parameters:

$m = 3 \, kg$, $K = 100 \, N/m$, $c = 3 \, Ns/m$,

Determine (a) damping factor, (b) natural frequency of damped vibration (c) logarithmic decrement (d) ratio of two consequtive amplitudes (e) no of cycles after which the original amplitude is reduced to 20%.

Different types of Damping:-

The damping is a physical system may be ~~done~~ one of the several types.

1. Viscous damping:-

- It is one of the most important type of damping and occurs for small velocities in ~~lubricating~~ lubricated sliding surfaces, dashpots, with small clearances. The amount of damping resistance will depend upon the relative velocity and upon the parameters of the damping system.
- One of the reasons for so much importance of this type of damping is that it affords an easy analysis of system by virtue of the fact that differential equation for the system become linear with this type of damping.

2. Dry friction or Coulomb damping:-

This type of damping occurs when two machine parts rub against each other, dry or unlubricated. The damping resistance in this case is practically constant and it is independent of the rubbing velocity.

3. Solid or structural damping:-

This type of damping is due to the internal friction of the molecules. The stress-strain diagram for a vibrating body is not a straight line but forms a hysterisis loop, the area of which represents the energy dissipated due to molecular friction per cycle per unit volume. The size of the loop depends upon the material of the vibrating body, frequency and amount of dynamic stress.

4. Slip or interfacial damping:-

Energy of vibration is dissipated by microscopic slip on the interfaces of m/c parts in contact under fluctuating loads. Microscopic slips also occurs on the interfaces of m/c elements forming various types of joints. The amount of damping depends amongst other things upon the surface roughness of the mating parts, the contact pressure and amplitude of vibration. It is a non-linear type damping.

Equations of free damped single dof system

Solution of the equation

$$\boxed{s_{1,2} = \left(-\frac{c}{2m}\right) \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}}}$$

Most general form of solution

$$\boxed{x = C_1 e^{s_1 t} + C_2 e^{s_2 t}}$$

Where C_1 and C_2 are two arbitrary constants to be determined from the initial conditions.

- A term critical damping coefficient, denoted by c_c
~~It is that value of the damping coefficient c that makes the expression~~ $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}}$ equals to zero

(i) over damped system ($\zeta > 1$)

$$s_1 = \left[-\zeta + \sqrt{\zeta^2 - 1}\right]\omega_n$$

$$s_2 = \left[-\zeta - \sqrt{\zeta^2 - 1}\right]\omega_n$$

Equation $$\boxed{x = C_1 e^{\left[-\zeta + \sqrt{\zeta^2-1}\right]\omega_n t} + C_2 e^{\left[-\zeta - \sqrt{\zeta^2-1}\right]\omega_n t}}$$

2. critically damped system ($\xi = 1$)

Roots $s_1 = s_2 = -\omega_n$

and. equation $\boxed{x = [c_1 + c_2 t]\, e^{-\omega_n t}}$

3. Underdamped system ($\xi < 1$)

$$\boxed{x = X e^{-\xi \omega_n t} \sin(\omega_d t + \phi)}$$

Assignment - 7 (P-73, Groover)

The mass of a spring mass dash pot system is given an initial velocity (from the equilibrium position) of $A\omega_n$ where ω_n is the undamped natural frequency of the system. Find the equation of motion of the system for cases when (i) $\xi = 2$, (ii) $\xi = 1$, (iii) $\xi = 0.2$.

Assignment - 8

The disc of a torsional pendulum has a moment of inertia of 600 kg cm2 and is immersed in a viscous fluid. The brass-shaft attached to it is of 10 cm dia and 40 cm long. When the pendulum is vibrating, it is observed amplitudes on the same side of the rest position for successive cycles are 9°, 6°, 4°. Determine

(a) logarithmic decrement

(b) damping torque at unit velocity

(c) periodic time of vibration.

Forced Vibration ~~with~~ of Single Degree of Freedom Systems

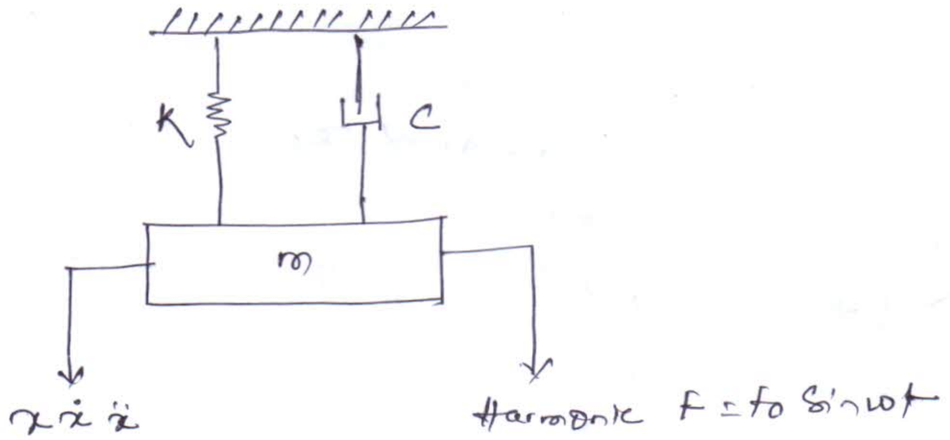
- In free vibration system, a system once disturbed from its equilibrium position, executes vibration because of it's elastic properties. The system will come to rest depending upon its damping characteristics.
- In case of forced vibration there is an impressed force on the system which keeps it vibrating.

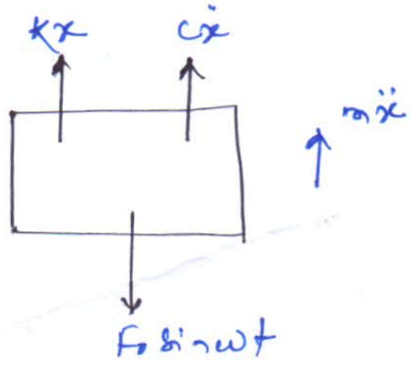
Examples:-

(a) air compressors
(b) Internal combustion engine
(c) machine tools and various other machineries.

Forced Vibration with Constant Harmonic Excitations:-

- In forced vibration the response of the system consists of two parts
  1. Transient and the system will vibrate with damped frequency
  2. steady state and the system will vibrate with the frequency of excitation.





From Newton's second law:

$$F_0 \sin\omega t - c\dot{x} - Kx - m\ddot{x} = 0$$

$$\Rightarrow m\ddot{x} + c\dot{x} + Kx = F_0 \sin\omega t \quad \text{---(1)}$$

Eq.(1) is a linear, second order differential equation and the solution has two parts.

- complementary function (transient part will dissapear)
- Particular integral

for complementary solution $m\ddot{x}+c\dot{x}+Kx=0$

- The particular solution is a steady state harmonic oscillation having a frequency equal to the excitation, and the displacement vector lags the force vector by some angle.

. Let the particular solution be $x_p = X\sin(\omega t-\phi)$ — (2)

where X = amplitude of vibration

$$\dot{x}_p = \omega X\cos(\omega t-\phi) = \omega X\sin(\omega t-\phi+\pi/2)$$

$$\ddot{x}_p = \omega^2 X\sin(\omega t-\phi+\pi)$$

As the complementary solution x_c will dissapear, we have

$$m\ddot{x}_p + c\dot{x}_p + Kx_p = F_0\sin\omega t$$

$$\Rightarrow F_0\sin\omega t - m\ddot{x}_p - c\dot{x}_p - Kx_p = 0 \quad \text{— (3)}$$

where $F_0\sin\omega t$ = impressed force

$m\ddot{x}_p$ = inertia force

Kx_p = spring force.

$c\dot{x}_p$ = damping force.

Substituting the values of $\ddot{x}_p$, $\dot{x}_p$ and x_p in eq. (3)

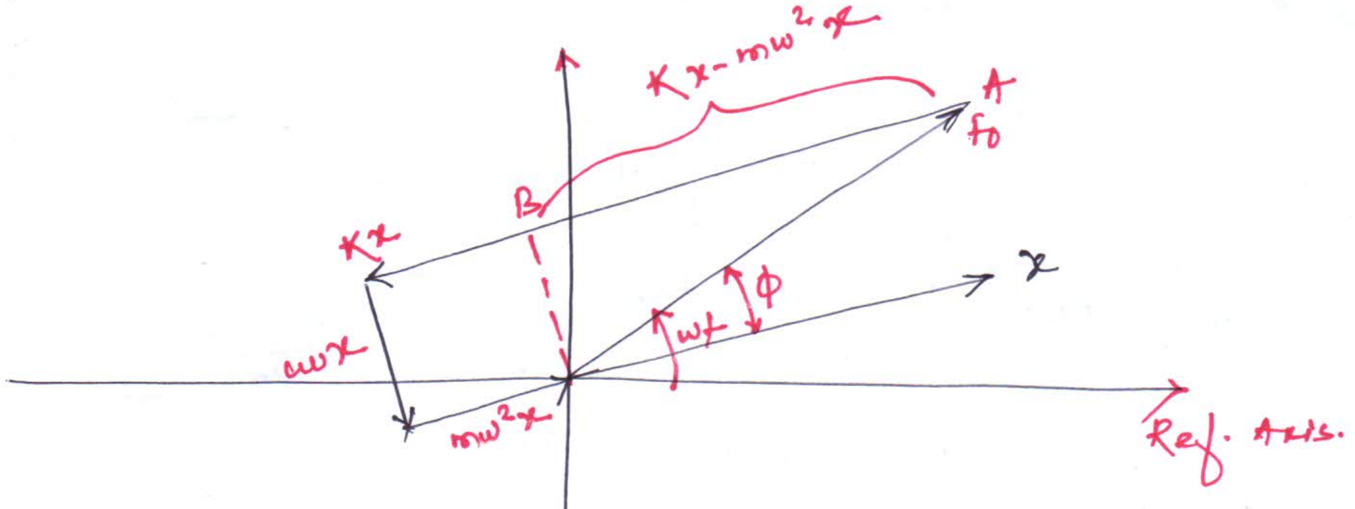
$$F_0\sin\omega t - m\omega^2 X\sin(\omega t-\phi+\pi) - c\omega X\sin(\omega t-\phi+\pi/2) - KX\sin(\omega t-\phi) = 0 \quad \text{— (4)}$$

The vectorial representation of equation (4) is as shown in the figure.

From the figure, we have $\tan\phi = \dfrac{c\omega X}{(KX - m\omega^2 X)}$

$$= \frac{c\omega}{(K-m\omega^2)} = \frac{c\omega/K}{\left(1-\frac{m\omega^2}{K}\right)}$$

$$= \frac{\frac{c}{c_c}\cdot\frac{c_c}{2m}\cdot\frac{2m\omega}{K}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)} = \frac{\xi\cdot\omega_n\cdot\frac{2\omega}{\omega_n^2}}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]} = \frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$



Now $\boxed{\phi = \tan^{-1} \dfrac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}}$ — (5)

when

$\xi = 0$	$\frac{\omega}{\omega_n} < 1$,	$\phi = 0$
ξ = any value,	$\frac{\omega}{\omega_n} = 1$	$\phi = \pi/2$
ξ = any value	$\frac{\omega}{\omega_n} \geq 1$	$\phi = \pi$

from the vectorial representation

$$f_0 = \sqrt{(Kx - m\omega^2 x)^2 + (c\omega x)^2}$$

$$\frac{f_0}{K} = X_{st} = X\sqrt{\left(\frac{K}{K} - \frac{m\omega^2}{K}\right)^2 + \left(\frac{c\omega}{K}\right)^2}$$

Where X_{st} = zero frequency deflection of the system

and $X_{st} = X\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + 2\xi\left(\frac{\omega}{\omega_n}\right)^2}$

and $\dfrac{X_{st}}{X} = \sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + 2\xi\left(\frac{\omega}{\omega_n}\right)^2}$

where $\dfrac{X}{X_{st}}$ = Magnification factor.

$$\Rightarrow \boxed{\frac{X}{X_{st}} = \frac{1}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}} \quad — (6)$$

At resonance $\boxed{\omega = \omega_n, \quad \dfrac{X}{X_{st}} = \dfrac{1}{2\xi}}$ = Magnification factor. — (7)

so the amplitude of vibration

$$X = \frac{X_{st}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad \text{--- (8)}$$

and phase lag $\phi = \tan^{-1}\left[\dfrac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$

We have the particular solution

$$x_p = X \sin(\omega t - \phi) \quad \text{--- (9)}$$

substituting the value of X in eq. (9) we have

$$\boxed{x_p = \frac{X_{st}\sin(\omega t - \phi)}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2}}} \quad \text{--- (10)}$$

In forced vibration $\left(\frac{\omega_p}{\omega_n}\right) = \sqrt{1-2\zeta^2}$ --- (11)

Also $\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$ --- (12)

where ω_p = frequency corresponding to the peak amplitude.

Example-1

A damped natural frequency of a system as obtained from a free vibration test is 9.8 Hz. During the forced vibration test with constant excitation force on the same system max amplitude of vibration is found to be 9.6 Hz. Find the damping factor for the system and it's natural frequency.

Given! $\omega_p = 9.6$ Hz $= (9.6 \times 2\pi)$ rad/sec.

$\omega_d = (9.8 \times 2\pi)$ rad/s.

we have the relation $\frac{\omega_p}{\omega_n} = \sqrt{1-2\zeta^2}$

$$\Rightarrow \frac{9.6 \times 2\pi}{\omega_n} = \sqrt{1-2\zeta^2}$$

also $\frac{\omega_d}{\omega_n} = \sqrt{1-\zeta^2}$

$$\Rightarrow \frac{9.8 \times 2\pi}{\omega_n} = \sqrt{1-\zeta^2}$$

Dividing the two equations

$$\frac{9.6}{9.8} = \frac{\sqrt{1-2\zeta^2}}{\sqrt{1-\zeta^2}}$$

$$\Rightarrow \boxed{\zeta = 0.196}$$

Substituting the value of $\zeta = 0.196$ in any of the two equations

$$\omega_n = (10 \times 2\pi) \text{ rad/s.}$$

$$\text{or } f_n = \frac{\omega_n}{2\pi} = 10 \text{ Hz} \quad \text{(Ans)}$$

Example - 2

Consider a spring-mass-damper system with $K = 4000$ N/m, $m = 10$ Kg and $c = 40$ N-s/m. Find the steady state and total response of the system under the harmonic force $F = 200 \sin 10t$ N for initial conditions $x = 0.1$ m and $\dot{x} = 0$, at $t = 0$.

Given:- $K = 4000$ N/m, $m = 10$ Kg, $c = 40$ N-s/m

$$\text{So } \omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s.}$$

$$\text{Now } \zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{40}{2\times10\times20} = 0.1$$

$$\omega_d = \sqrt{1-\zeta^2}\cdot\omega_n = \sqrt{1-0.1^2} \times 20 = 19.9 \text{ rad/s.}$$

Steady state amplitude

$$X = \frac{X_{st}}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2}}$$

$$\Rightarrow X = \frac{f_0/K}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\zeta\frac{\omega}{\omega_n}\right\}^2}}$$

$$= \frac{200/4000}{\sqrt{\left\{1-\left(\frac{10}{20}\right)^2\right\}^2 + \left(\frac{2\times0.1\times10}{20}\right)^2}}$$

$$= 0.066 \text{ m.}$$

$$\text{phase lag } \phi = \tan^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] = \tan^{-1}\left[\frac{2\times0.1\times(10/20)}{\left\{1-\left(\frac{10}{20}\right)^2\right\}}\right]$$

$$= 7.59^\circ$$

The steady state response of the system is given by

$x_p = X \sin(\omega t - \phi)$

$= 0.066 \sin(10t - 7.59^\circ)$

The transient response $x_c = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1)$

Total response of the system

$x = x_c + x_p$

$= Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) + 0.066 \sin(10t - 7.59^\circ)$ — (1)

The values of A and ϕ_1 are calculated from the initial conditions.

Now differentiating eq. (1) we have

$\dot{x} = -\zeta\omega_n Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) + A\omega_d e^{-\zeta\omega_n t} \cos(\omega_d t + \phi_1)$

$+ .066 \times 10 \cos(10t - 7.59^\circ)$ — (2)

substituting the initial conditions

$0.1 = A \sin\phi_1 + 0.066 \sin(-7.59^\circ)$

$\Rightarrow 0.1 = A \sin\phi_1 - 0.0087$

$\Rightarrow \boxed{A \sin\phi_1 = 0.1087}$ — (3)

from eq. (2) $0 = -\zeta\omega_n A \sin\phi_1 + A\omega_d \cos\phi_1 + 0.654$

$\Rightarrow \boxed{A\cos\phi_1 = -0.020}$ — (4)

$\tan\phi_1 = \dfrac{0.1087}{-0.020} \Rightarrow \boxed{\phi_1 = -79.57^\circ}$

and $\boxed{A = 0.11}$

so the total response of the system is given by

$x = 0.11 e^{-2t} \sin(19.9t - 79.57^\circ) + 0.066 \sin(10t - 7.59^\circ)$ (Ans).

Example 3

find the natural ~~frequency~~ response of a single dof system with $m = 10$ kg, $c = 50$ N-s/m, $k = 2000$ N/m under the action of harmonic force $F = F_0 \sin\omega t$ with $F_0 = 200$ N and $\omega = 31.416$ rad/s. The initial conditions may be assumed as $x = 0.01$ m and $\dot{x} = 5\,m/s$ at $t = 0$.

from the given data $\omega_n = \sqrt{\frac{K}{m}} = 14.142\ rad/s$.

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{50}{2\times 10\times 14.142} = 0.1768$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 13.92\ rad/s.$$

$$X_{st} = \frac{F_0}{K} = \frac{200}{2000} = 0.1\ m.$$

steady state amplitude

$$X = \frac{X_{st}}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\zeta\frac{\omega}{\omega_n}\right\}^2}} = \frac{0.1}{\sqrt{\left\{1-\left(\frac{31.416}{14.142}\right)^2\right\}^2 + \left(\frac{2\times .1768\times 31.416}{14.142}\right)^2}}$$

$$\Rightarrow \boxed{X = 0.0249\ m.}$$

$$\phi = \tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$= -11.53^\circ$$

Total response of the system is given by

$$x = x_c + x_p$$

$$= Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi_1) + X\sin(\omega t \mp \phi)$$

$$= A e^{-2.5t}\sin(13.92t + \phi_1) + 0.0249\sin(31.416t + 11.53^\circ) \quad -(1)$$

Differentiating eq. (1) wrt time

$$\dot{x} = -2.5Ae^{-2.5t}\sin(13.92t + \phi_1) + 13.92Ae^{-2.5t}\cos(13.92t + \phi_1)$$
$$+ (0.0249)(31.416)\cos(31.416t + 11.53^\circ) \quad -(2)$$

Applying initial conditions we have

$$0.01 = A\sin\phi_1 + 0.0249\sin 11.53^\circ$$

$$0.01 = A\sin\phi_1 + 0.0049$$

$$\Rightarrow \boxed{A\sin\phi_1 = 0.005} \quad -(3)$$

and $5 = -2.5A\sin\phi_1 + 13.92A\cos\phi_1 + 0.766$

$$\Rightarrow \boxed{A\cos\phi_1 = 0.305} \quad -(4)$$

$$\boxed{\phi_1 = 0.94^\circ}$$

$$A = \cancel{\tan^{-1}\left(\frac{0.005}{0.305}\right)} = \boxed{0.3}$$

The total response

$$x = 0.3\, e^{-2.15t} \sin(13.92t + 0.94^{\circ}) + 0.0249 \sin(31.416t + 11.53^{\circ})$$

Example - 4

find out the frequency ratio for which amplitude in forced vibration will be maximum. Also determine the peak amplitude and the corresponding phase angle.

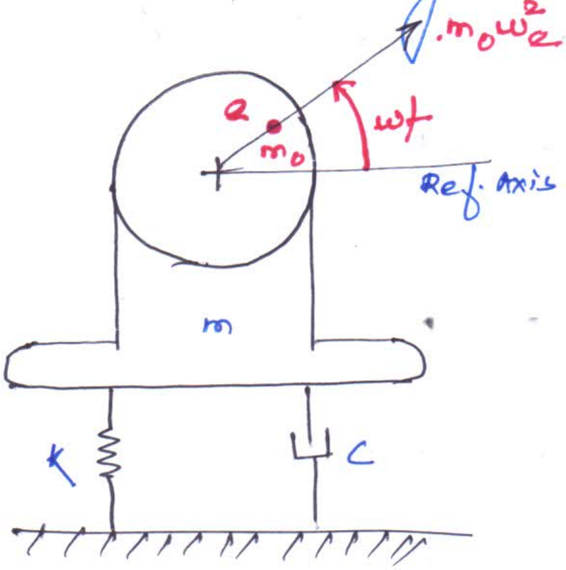
Forced Vibration with Rotating and Reciprocating Unbalance:

All rotating machinery like electric motor, turbine etc. have some amount of unbalance left in them after correcting their unbalance on precession balancing m/c.

Let m_0 = an equivalent mass rotating with it's centre of gravity 'e' from axis of rotation.

- Then the final unbalance is measured in terms of the equivalent mass m_0 rotating with it's centre of gravity at a distance 'e' from the axis of rotation.

The centrifugal force generated because of the rotation of the body is proportional to the square of the frequency of rotation. This force is max^m value of the sinusoidal excitation in any direction.



Consider an elastically supported m/c rotating at 'ω' rad/s. Let the unbalance mass m_0 have an eccentricity 'e'.

Let m = total mass of the m/c including m_0

k = spring stiffness

c = damping coefficient

Let m_0 mass makes an angle ωt with the reference axis at any instant.

The equation of motion in vertical axis is:

$$(m-m_0)\frac{d^2x}{dt^2} + m_0\frac{d^2}{dt^2}(x+e\sin\omega t) = -Kx - c\dot{x}$$

or $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = m_0 e\omega^2 \sin\omega t \quad \text{— (1)}$$

Comparing Eq.(1) with that of the eq. of motion for a forced vibration of single dof system

f_0 is replaced by $m_0 e \omega^2$

Therefore the steady state amplitude is given by

$$\boxed{X = \frac{m_0 e \omega^2 / k}{\sqrt{\left(1 - \frac{m\omega^2}{k}\right)^2 + \left(\frac{c\omega}{k}\right)^2}}} \quad \text{--- (2)}$$

In a dimensionless form

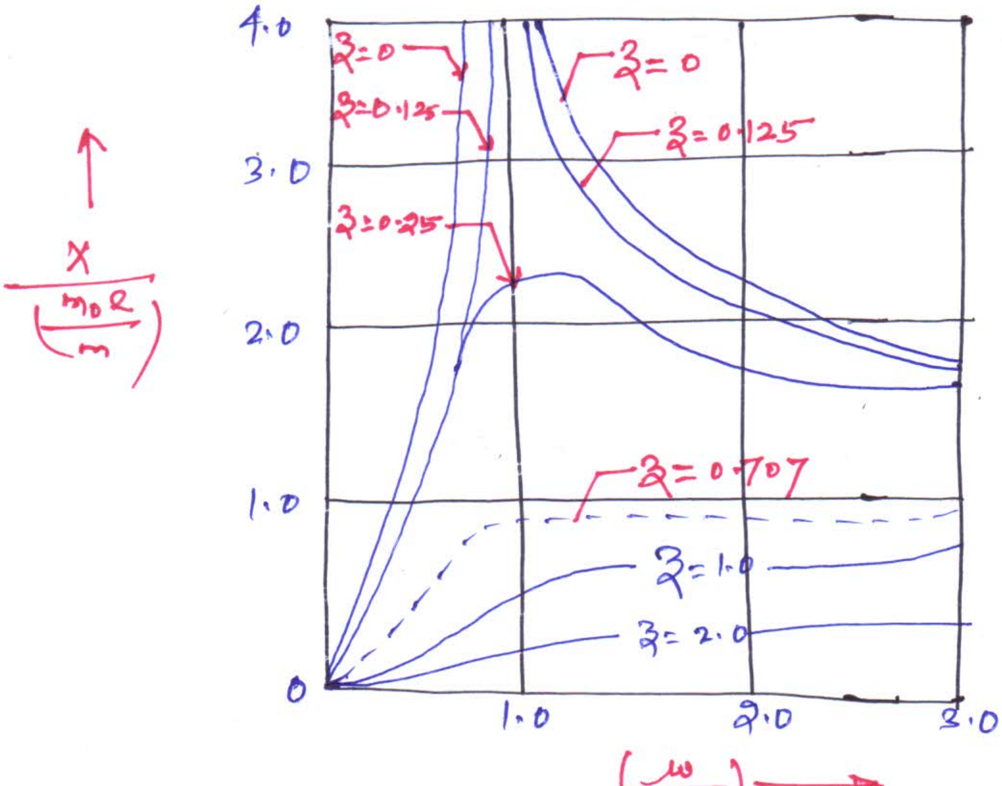
$$\boxed{\frac{X}{\left(\frac{m_0 e}{m}\right)} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}} \quad \text{--- (3)}$$

Phase lag

$$\boxed{\phi = \tan^{-1}\left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]} \quad \text{--- (4)}$$

At low speed the centrifugal exciting force $m_0 e \omega^2$ is small and therefore the response curve starts from zero. At resonance $\omega/\omega_n = 1$ and

$$\boxed{\frac{X}{\left(\frac{m_0 e}{m}\right)} = \frac{1}{2\xi}} \quad \text{--- (5)}$$



(Dimensionless amplitude Vs frequency ratio plot)

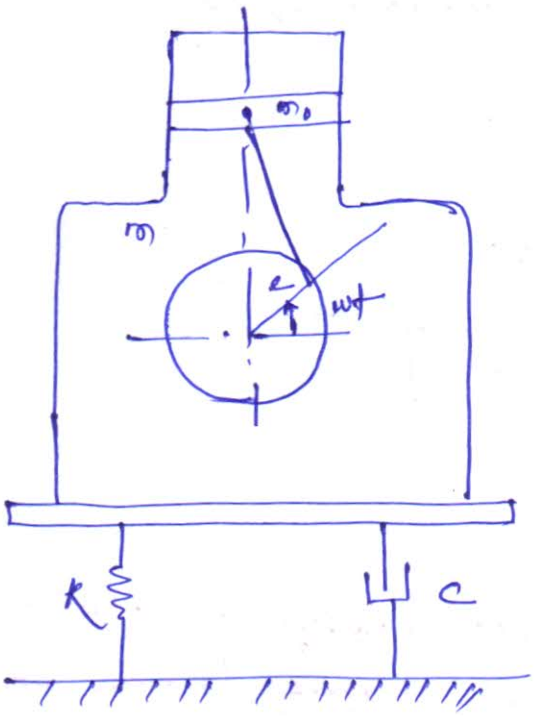
Vibration anlysis of Reciprocating mass:-

Let m_0 = equivalent mass of reciprocating part

m = total mass of the engine including the reciprocating mass.

e = crank length

l = length of connecting rod.



The inertia force due to the reciprocating mass is approximately

$$= m_0 e w^2 \left\{ \sin wt + \left(\frac{e}{l}\right) \sin 2wt \right\}$$

If 'e' is small compared to l, the second harmonic may be neglected and the exciting force becomes equal to $m_0 e w^2 \sin wt$ and is same as in case of rotating unbalance mass.

Therefore for small 'e' same vibration analysis is followed in case of reciprocating unbalance mass.

Example-1

A system of beam supports a motor of mass 1200 kg. The motor has an unbalanced mass of 1 kg located at 6 cm radius. It is known that resonance occurs at 2210 rpm. What amplitude of vibration can be expected at motor's operating speed of 1440 rpm if damping factor is 0.1 and 0 respectively.

We have $\frac{\omega}{\omega_n} = \frac{1440}{2210} = 0.652$

$\frac{m_0}{m} = \frac{1}{1200}$, $e = 0.06\ m$

$\zeta = 0.1$

Using the relation

$$\frac{x}{\left(\frac{.06}{1200}\right)} = \frac{(0.652)^2}{\sqrt{\{1-0.652^2\}^2 + \{2 \times 0.1 \times 0.652\}^2}}$$

$\Rightarrow \boxed{x = 0.036\ mm}$

If $\zeta = 0$

$$\frac{x}{\left(\frac{.06}{1200}\right)} = \frac{(0.652)^2}{[1-0.652^2]}$$

$\Rightarrow \boxed{x = 0.037\ mm}$

Example-2

A single cylinder vertical petrol engine of total mass 320 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 0.2 cm. The reciprocating parts of the engine have a mass of 24 kg. and move through a vertical stroke of 15 cm with SHM. A dashpot is provided, the damping resistance of which is directly proportional to the velocity and amounts to 490 N at 0.3 m/s. Determine

(a) the speed of driving shaft at which resonance will occur

(b) amplitude of steady state forced vibration when the driving shaft of the engine rotates at 480 rpm.

Let $m = 320$ Kg $\Delta st = 0.002$ m $m_0 = 24$ Kg.

$$e = \frac{0.15}{2} = 0.075 \text{ m}.$$

$$\omega_n = \sqrt{\frac{g}{\Delta st}} = \sqrt{\frac{9.81}{0.002}} = 70 \text{ rad/s}.$$

$$\text{resonance speed} = \frac{70}{2\pi} \times 60 = 670 \text{ rpm}$$

$$\omega = \frac{480 \times 2\pi}{60} = 50.4 \text{ rad/sec}.$$

$$\text{so } \left(\frac{\omega}{\omega_n}\right) = \frac{50.4}{70} = 0.72$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{490/0.3}{2 \times 320 \times 70} = 0.0364$$

$$\frac{m_0}{m} = \frac{24}{320} = 0.075$$

$$\text{Now } \frac{X}{\left(\frac{m_0 e}{m}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\Rightarrow \frac{X}{0.075 \times .075} = \frac{(0.72)^2}{\sqrt{(1-.72)^2 + (2 \times .0364 \times .72)^2}}$$

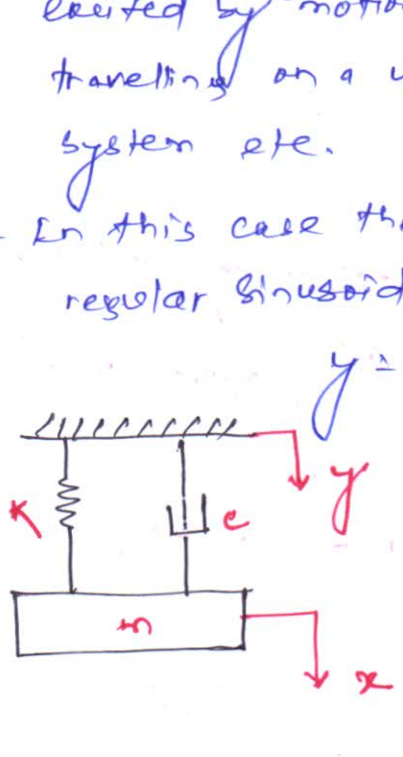
$$\Rightarrow \boxed{X = 0.006 \text{ m or } 6 \text{ mm}}$$

forced vibration due to base excitation:-

In most of the vibration related problems, a system is being excited by motion of the support. for example a vehicle is travelling on a wavy road, an engine mounted on a vibrating system etc.

- In this case the support is considered to be excited by a regular sinusoidal motion,

$$y = Y \sin \omega t \quad \text{---(1)}$$



considering a spring-mass-damper system the mass is attached with the support by means of a spring of stiffness K, a damper of damping coefficient c.

Absolute Amplitude:-

Let x = absolute motion of mass m.

Equation of motion for the system may be written as:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + K(x - y) = 0$$

or $$m\ddot{x} + c\dot{x} + Kx = ky + c\dot{y} \quad \text{--- (2)}$$

We have $y = Y \sin \omega t$

or $\dot{y} = \omega Y \cos \omega t$

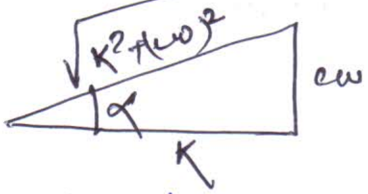
Substituting the value of y and $\dot{y}$ in eq. (2), we have

$$m\ddot{x} + c\dot{x} + Kx = KY \sin \omega t + c\omega Y \cos \omega t$$

or $$m\ddot{x} + c\dot{x} + Kx = Y\left[K \sin \omega t + c\omega \cos \omega t\right]$$

or $$m\ddot{x} + c\dot{x} + Kx = Y\sqrt{K^2 + (c\omega)^2}\left[\frac{K}{\sqrt{K^2 + (c\omega)^2}} \sin \omega t + \frac{c\omega}{\sqrt{K^2 + (c\omega)^2}} \cos \omega t\right]$$

or $$m\ddot{x} + c\dot{x} + Kx = Y\sqrt{K^2 + (c\omega)^2}\left[\cos\alpha \sin \omega t + \sin\alpha \cos \omega t\right]$$



or $$m\ddot{x} + c\dot{x} + Kx = Y\sqrt{K^2 + (c\omega)^2} \sin(\omega t + \alpha) \quad \text{--- (3)}$$

where $$\alpha = \tan^{-1}\left(\frac{c\omega}{K}\right) = \tan^{-1}\left(2\xi\frac{\omega}{\omega_n}\right) \quad \text{--- (4)}$$

Equation (3) is same as that of the equation of forced vibration with harmonic excitation

$$m\ddot{x} + c\dot{x} + Kx = F_0 \sin \omega t.$$

Therefore the steady-state solution of eq. (3) is

$$x = X \sin(\omega t - \phi) \quad \text{--- (5)}$$

where X = steady state amplitude

and $$X = \frac{Y\sqrt{K^2 + (c\omega)^2}}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}}$$

$\left[\because \text{ same as forced vib. eq. } X = \dfrac{F_0}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}}\right]$

In a dimensionless form

$$\boxed{\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}} \quad \text{--- (6)}$$

and $$\boxed{\phi = \tan^{-1}\left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]} \quad \text{--- (7)}$$

Comparing Eq. (1) and (5), it can be seen that the motion of mass 'm' lags that of the support through an angle $(\phi - \alpha)$.

Therefore the angle of lag $(\phi - \alpha)$.

$$\boxed{= \tan^{-1}\left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left[2\xi\frac{\omega}{\omega_n}\right]} \quad \text{---} (8)$$

Equation (5), (6) and (8) completely define the absolute motion of mass 'm' because of base excitation.

Relative amplitude :-

Let z = relative motion of mass m w.r.t the support

then $z = x - y$

or $x = y + z$

We have the eq. of motion of mass for an absolute amplitude case is

$$m\ddot{x} + c\dot{x} + Kx = Ky + c\dot{y}$$

or $\quad m(\ddot{y}+\ddot{z}) + c(\dot{y}+\dot{z}) + K(y+z) = Ky + c\dot{y}$

or $\quad m\ddot{z} + c\dot{z} + Kz = -m\ddot{y} \quad \text{---} (9)$

The base is excited by a regular sinusoidal equation

$y = Y\sin\omega t$

so $\quad \dot{y} = \omega Y\cos\omega t$

$\ddot{y} = -\omega^2 Y\sin\omega t$

Substituting the value of $\ddot{y}$ in eq. (9)

$$m\ddot{z} + c\dot{z} + Kz = m\omega^2 Y\sin\omega t \quad \text{---} (10)$$

Eq. (10) is same as that of equation of forced vibration with rotating unbalance

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = m_0 e\omega^2\sin\omega t$$

with a solution $\quad X = \dfrac{(m_0 e\omega^2/K)}{\sqrt{\left(1-\frac{m\omega^2}{K}\right) + \left(\frac{c\omega}{K}\right)^2}}$

and therefore the solution in a dimensionless form

$$\frac{Y}{Z} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad \text{——— (11)}$$

and $$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \quad \text{——— (12)}$$

Example-3

The support of a spring-mass system is vibrating with an amplitude of 5mm and a frequency of 1150 cycle/min. If the mass is 0.9 kg and spring stiffness of 1960 N/m, determine the amplitude of vibration of mass. What amplitude will result if a damping factor of 0.2 is included in the system?

Given data:-

Mass $m = 0.9$ Kg, $Y = 5$ mm $\quad K = 1960$ N/m

frequency = 1150 cycle/min = $1150 \times \frac{2\pi}{60} = 120.3$ rad/s

Now $\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{1960}{0.9}} = 46.7$ rad/sec.

$$\frac{\omega}{\omega_n} = \frac{120.3}{46.7} = 2.58$$

The equation for base excitation for absolute amplitude

$$\frac{X}{Y} = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

for $\xi = 0$

$$\frac{X}{5} = \left|\frac{1}{[1-2.58^2]}\right| = \frac{1}{5.65}$$

$$\Rightarrow \boxed{X = 0.886\text{ mm}}$$

for $\xi = 0.2$

$$\frac{X}{5} = \frac{\sqrt{1+(2\times0.2\times2.58)^2}}{\sqrt{(1-2.58^2)^2+(2\times0.2\times2.58)^2}} =$$

$$\Rightarrow \boxed{X = 1.25\text{ mm}}$$ (Ans)

Observation from the $\frac{X}{\left(\frac{m_0 e}{m}\right)}$ vs $\left(\frac{\omega}{\omega_n}\right)$ plot :-

1. All the curves begin at zero amplitude.
2. At resonance, the amplitude of vibration is given by
   $\frac{X}{\left(\frac{m_0 e}{m}\right)} = \frac{1}{2\zeta}$, which indicates that the damping factor plays important role in controlling the vibration amplitude at resonance.
3. At very high speeds, $\frac{X}{\left(\frac{m_0 e}{m}\right)}$ tends to unity and damping has negligible effect.
4. for $0 < \zeta < \frac{1}{\sqrt{2}}$, the peak occur to the right of the resonance value of $\frac{\omega}{\omega_n} = 1$.

Vibration Isolation and Transmissibility :-

Most of the machines when mounted or installed on the foundations, cause undesirable vibrations because of unbalanced forces set-up during their running. The vibration of large amplitude may damage the structure on which machines are mounted.

- Examples of these undesirable vibration cases are:
  - inertia forces developed in reciprocating engine
  - unbalanced force produced in any rotating m/c etc.

The effectiveness of isolation may be measured in terms of the ratio of force or motion transmitted to that in existence. The first type is called
- force isolation and the second one is called
- motion isolation.

- The lesser the force or motion transmitted, the greater is said to be the isolation.
- for isolation different materials are used such as

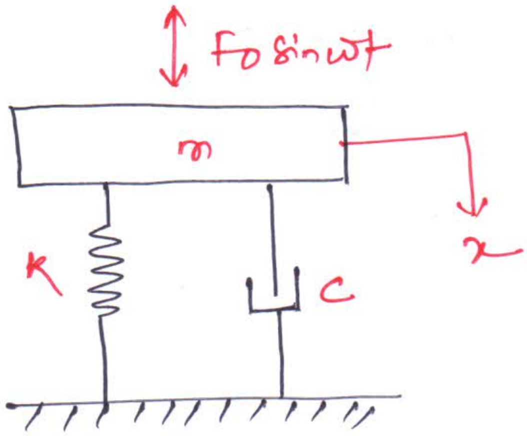
- pads of rubber
- felt or cork
- metallic spring etc.

- All these isolating materials are elastic and have damping properties.

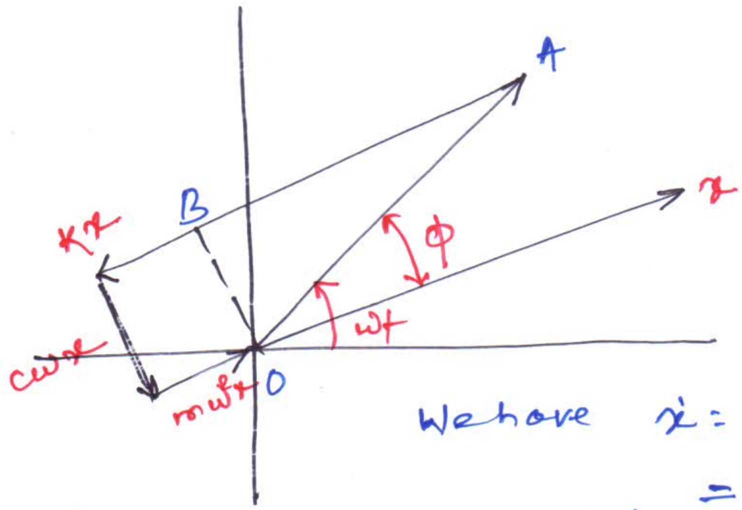
Force Transmissibility :-

Force transmissibility is defined as the ratio of

$$\boxed{\frac{\text{force transmitted to the foundation}}{\text{force impressed on the system.}}}$$



considering a case, where a mass m is supported on the foundation by means of an isolator having equivalent stiffness and damping coefficient of K and C respectively. The system is excited by a force $= F_0 \sin\omega t$



The differential equation of motion is

$$m\ddot{x} + c\dot{x} + Kx = F_0 \sin\omega t \quad \text{---(1)}$$

Assuming a particular solution of eq.(1)

$$x = X \sin(\omega t - \phi) \quad \text{---(2)}$$

We have $\dot{x} = \omega X \cos(\omega t - \phi)$

$$= \omega X \sin(\omega t - \phi + \pi/2) \quad \text{---(3)}$$

And $\ddot{x} = -\omega^2 X \sin(\omega t - \phi)$

$$= \omega^2 X \sin(\omega t - \phi + \pi) \quad \text{---(4)}$$

substituting the value of x, $\dot{x}$ and $\ddot{x}$ in eq. (1)

$$m\omega^2 X \sin(\omega t - \phi + \pi) + c\omega X \sin(\omega t - \phi + \pi/2) + KX \sin(\omega t - \phi) = F_0 \sin \omega t$$

or $$F_0 \sin\omega t - KX \sin(\omega t - \phi) - c\omega X \sin(\omega t - \phi + \pi/2) - m\omega^2 X \sin(\omega t - \phi + \pi) = 0 \quad \text{— (5)}$$

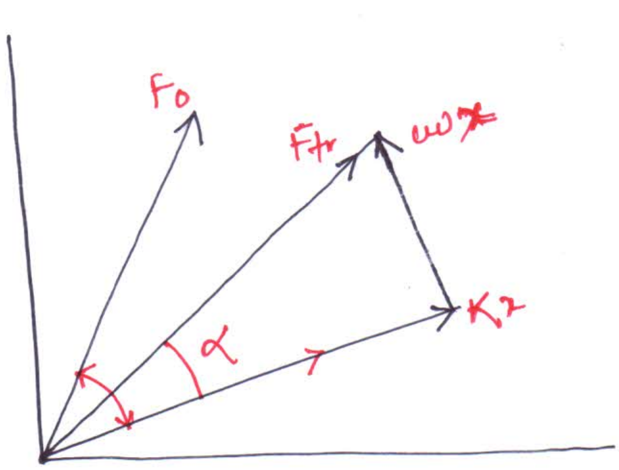
Total forces acting on the system are

1. External excitation force
2. Spring force
3. Dashpot force.
4. Inertial force.

Out of these four forces, the spring force KX and dashpot force $c\omega X$ are two common forces acting on the mass and on the foundation. Therefore the force transmitted to the foundation is the vector sum of these two forces.

Therefore $$F_{tr} = \sqrt{(KX)^2 + (c\omega X)^2}$$

$$\Rightarrow F_{tr} = X\sqrt{K^2 + (c\omega)^2} \quad \text{— (6)}$$



From the vector diagram, to find the value of X and ϕ in eq. (2) consider a $\perp$r triangle OAB by dropping OB to AB

Now $$F_0 = \sqrt{(KX - m\omega^2 X)^2 + (c\omega X)^2}$$
$$= X\sqrt{(K - m\omega^2)^2 + (c\omega)^2}$$

$$\Rightarrow X = \frac{F_0}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \quad \text{— (7)}$$

and $$\phi = \tan^{-1}\left[\frac{c\omega}{K - m\omega^2}\right] \quad \text{— (8)}$$

substituting the value of X in eq. (6)

force transmitted $F_{tr} = \dfrac{F_0\sqrt{K^2+(c\omega)^2}}{\sqrt{(K-m\omega^2)^2+(c\omega)^2}}$ — (9)

Eq. (9) can be represented as a dimensionless form

transmissibility $\boxed{T_r = \dfrac{F_{tr}}{F_0} = \dfrac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}}$ — (10)

The angle through which the transmitted force lags the impressed force is $(\phi-\alpha)$

where $\alpha = \tan^{-1}\left(\dfrac{c\omega X}{KX}\right) = \tan^{-1}\left(\dfrac{c\omega}{K}\right)$

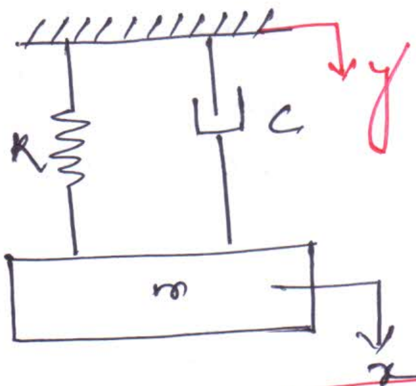
$\Rightarrow \alpha = \tan^{-1}\left(2\xi\dfrac{\omega}{\omega_n}\right)$

and angle $\phi = \tan^{-1}\left[\dfrac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$

so phase lag $\boxed{\phi-\alpha = \tan^{-1}\left[\dfrac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left(2\xi\dfrac{\omega}{\omega_n}\right)}$

— (11)

Motion Transmissibility :-



Motion transmissibility

$\boxed{\dfrac{X}{Y} = \dfrac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}}$ — (12)

$\phi = \tan^{-1}\left[\dfrac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$ — (13)

Phase lag $\boxed{\phi-\alpha = \tan^{-1}\left[\dfrac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left[2\xi\dfrac{\omega}{\omega_n}\right]}$ — (14)

Typical Isolators used:-

- coil springs
- elastometers (rubber and neoprene)

Coil springs / steel springs:-

These are generally used for $f_n < 6\,Hz$ and $\Delta st > 7.5\,mm$
Large coil diameter is chosen for larger deflection

Pad Mounts:-

Ribbed neoprene mounts are used for small static deflection. They can be used in a series for a total maximum static deflection of about 4mm. They are generally used for printing machinery, saws, transformers, vaccum pumps wood working machinery etc.

General purpose Elastometric mounts:-

They are used in compression/shear, for static deflection from 2mm to 16mm corresponding to natural frequencies from 11 Hz to 4 HZ. They are used with great variety of machines including blowers, fans, pumps, bending m/cs diesel engine, motor generator sets etc.

Example-1

A 1000 Kg machine is mounted on four identical springs of total spring constant K and having negligible damping. The machine is subjected to a harmonic external force of amplitude $F_0 = 490\,N$ and frequency 180 rpm. Determine (a) the amplitude of motion of the machine and maximum force transmitted to foundation because of the unbalanced force when $K = 1.96 \times 10^6\,N/m$. (b) the same as in (a) for the case when $K = 9.8 \times 10^4\,N/m$

(a) $K = 1.96 \times 10^6$ N/m $m = 1000$ kg.

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{1.96 \times 10^6}{1000}} = 44.3 \text{ rad/sec.}$$

$$\frac{\omega}{\omega_n} = \frac{180 \times 2\pi}{60 \times 44.3} = 0.425$$

$F_0 = 490$ N $\zeta = 0$

$$X_{st} = \frac{F_0}{K} = \frac{490}{1.96 \times 10^6} = 2.5 \times 10^{-4} \text{ m.}$$

Amplitude

$$\frac{X}{(X_{st})} = \frac{1}{\sqrt{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\} + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

As $\zeta = 0$

$$\frac{X}{(X_{st})} = \left|\frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right| = \frac{1}{|1 - 0.425^2|} = \boxed{0.305 \text{ mm}}$$

Transmitted force

$$\frac{F_{tr}}{490} = \frac{1}{(1 - 0.425^2)} \Rightarrow F_{tr} = \frac{490}{0.819} = \boxed{607 \text{ N.}}$$

(b) $K = 9.8 \times 10^4$ N/m. $m = 1000$ kg.

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{9.8 \times 10^4}{1000}} = 9.9 \text{ rad/sec.}$$

$$\frac{\omega}{\omega_n} = \frac{180 \times 2\pi}{60 \times 9.9} = 1.90$$

$F_0 = 490$ N, $\zeta = 0$

$$X_{st} = \frac{F_0}{K} = \frac{490}{9.8 \times 10^4} = 0.005 \text{ m.}$$

Now using the relation

$$\frac{X}{(X_{st})} = \frac{1}{|1 - 1.90^2|} = \frac{1}{2.61}$$

$\Rightarrow$ Amplitude $\boxed{X = 1.9 \text{ mm}}$

Transmitted force $\frac{F_{tr}}{490} = \frac{1}{|1 - 1.90^2|} \Rightarrow F_{tr} = \frac{490}{2.61} = \boxed{188 \text{ N.}}$

Example-2

A 75Kg machine is mounted on springs of stiffness K = 11.76×10^5 N/m with an assumed damping factor of $\xi = 0.20$. A 2Kg piston within the machine has a reciprocating motion with a stroke of 0.08m and a speed of 3000 rpm. Assuming the motion of the piston to be harmonic, determine the amplitude of vibration of the machine and the vibrating force transmitted to the foundation.

Given data:-

mass of m/c $m = 75$ Kg. spring stiffness $K = 11.76\times10^5$ N/m

damping factor $\xi = 0.2$ equivalent unbalanced mass $m_0 = 2$ Kg.

$$e = \frac{0.08}{2} = 0.04\ \text{m} \qquad \text{speed} = 3000\ \text{rpm}$$

$$\omega = \frac{3000\times2\pi}{60} = 100\pi\ \text{rad/sec.}$$

$$\left(\frac{\omega}{\omega_n}\right) = \left(\frac{100\pi}{\omega_n}\right)$$

$$\text{Now } \omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{11.76\times10^5}{75}} = 125\ \text{rad/sec.}$$

$$\text{so } \left(\frac{\omega}{\omega_n}\right) = \frac{100\pi}{125} = 2.51$$

$$\frac{m_0 e}{m} = \frac{2\times0.04}{75} = 10.67\times10^{-4}\ \text{m.}$$

$$F_0 = m_0 e \omega^2 = 2\times0.04\times(100\pi)^2 = 7900\ \text{N.}$$

Now using the relation

$$\frac{X}{\left(\frac{m_0 e}{m}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$\Rightarrow \frac{X}{10.67\times10^{-4}} = \frac{2.51^2}{\sqrt{(1-2.51^2)^2 + (2\times0.2\times2.51)^2}}$$

$$\Rightarrow \boxed{X = 1.25\ \text{mm}}$$

Transmitted force

We have

$$\frac{F_{Tr}}{F_0} = \frac{\sqrt{1+\{2\zeta\left(\frac{\omega}{\omega_n}\right)\}^2}}{\sqrt{\{1-\left(\frac{\omega}{\omega_n}\right)^2\}^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\Rightarrow \frac{F_{Tr}}{7900} = \frac{\sqrt{1+\left(2\times0.2\times2.51\right)^2}}{\sqrt{\left(1-2.57^2\right)^2+\left(2\times0.2\times2.51\right)^2}}$$

$$\Rightarrow \boxed{F_{Tr} = 2078N.}$$ (Ans).

Example. 3

A radio set of 20kg mass must be isoted from a machine vibrating with an amplitude of 0.05 mm at 500 rpm. A set is mounted on four isolators, each having a spring scale of 31400 N/m and damping factor of 392 N-s/m

(a) What is the amplitude of vibration of the radio?

(b) What is the dynamic load on each isolator due to vibration?

Let m ~~___~~ = mass of radio set

K = equivalent spring stiffness

c = damping coefficient of the four isolator

$m = 20kg$ $K = 4\times31400 = 125600$ N/m

$C = 4\times392 = 1568$ NS/m

$y = Y\sin\omega t$ and $Y = 0.05$ mm

$$\omega = \frac{2\pi N}{60} = \frac{2\pi\times500}{60} = 52.5 \text{ rad/sec.}$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{125600}{20}} = 79.2 \text{ rad/sec.}$$

$$\left(\frac{\omega}{\omega_n}\right) = \frac{52.5}{79.2} = 0.662$$

$$\zeta = \frac{c}{2\sqrt{Km}} = \frac{1568}{2\sqrt{125600\times20}} = \boxed{0.496}$$

(a) Amplitude of vibration of radio set

$$\frac{X}{Y} = \frac{\sqrt{1+\left(2\times\zeta\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\{2\times\zeta\times\left(\frac{\omega}{\omega_n}\right)\}^2}}$$

$$\frac{X}{0.05} = \frac{\sqrt{1+(2\times 0.496\times 0.662)^2}}{\sqrt{(1-0.662)^2+(2\times 0.496\times .662)^2}}$$

$$\Rightarrow \boxed{X = 0.069\,mm} \quad (Ans)$$

(b) The dynamic load on isolators due to vibration can be obtained by finding the relative amplitude and then

$$F_{Dyn} = Z\sqrt{K^2+(c\omega)^2}$$

Now using the relation

$$\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\times\xi\times\frac{\omega}{\omega_n}\right)^2}}$$

$$\Rightarrow \frac{Z}{0.05} = \frac{(0.662)^2}{\sqrt{(1-0.662^2)^2+(2\times 0.496\times 0.662)^2}}$$

$\Rightarrow Z = 0.025\,mm$ or $2.5\times 10^{-5}\,m$

Now

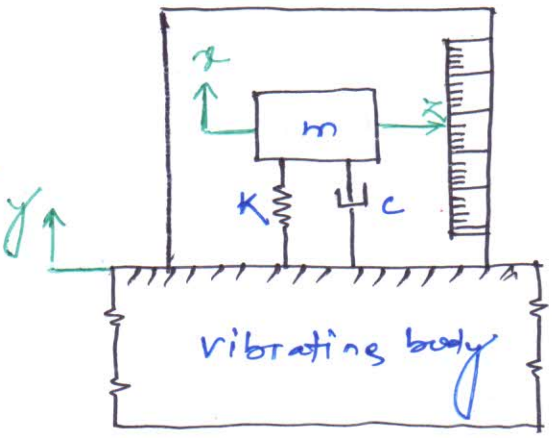
$$F_{Dyn} = Z\sqrt{K^2+(c\omega)^2}$$
$$= 2.5\times 10^{-5}\sqrt{125600^2+(1568\times 52.5)^2}$$
$$= 3.77\,N.$$

So the dynamic load on each isolator $= \frac{3.77}{4} = 0.94\,N.$

Vibration Measuring Instruments:-

The instrument used to measure any of the vibration related phenomenon i.e. displacement, velocity or acceleration of a vibrating system are referred to as vibration measuring instrument

The basic elements of most of the vibration measuring instrument is the seismic unit shown in the figure.



- It consists of a seismic mass m mounted on a spring K and dashpot c inside a box. The box is then placed on the vibrating machine or body.

The arrangement is similar to the spring-mass-dashpot system having support. The displacement of the mass relat to the box i.e. 'Z' can be measured by attaching a pointer to the mass and a scale to the box.

Vibrometer :- (Displacement Measuring Instrument) :

Vibrometer is a used to measure the displacement of a vibrating body.

Considering the equation

$$\boxed{\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left\{2\xi\left(\frac{\omega}{\omega_n}\right)\right\}^2}}} \quad \text{—— (1)}$$

When the natural frequency of the instrument is low in comparision to vibrating frequency ω, the relative displacement approaches the amplitude of vibrating body irrespective of damping in the instrument

If $\frac{\omega}{\omega_n} >> 1$, then eq. (1) may be written as

$$\boxed{\frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2}} \simeq 1} \qquad \boxed{\because 1-\left(\frac{\omega}{\omega_n}\right)^2 \simeq \left(\frac{\omega}{\omega_n}\right)^2 \text{ for } \frac{\omega}{\omega_n} >> 1}$$

$$\Rightarrow \boxed{Z \simeq Y} \quad \text{—— (2)}$$

- Thus, when $\frac{\omega}{\omega_n}$ is large, amplitude recorded is approximately equal to the amplitude of vibrating body. In most of the vibrometers, damping is kept as small as possible.

- Vibrometer are therefore known as low natural frequency instruments. The average value of natural frequency, ω_n for vibrometer is about 4 Hz.

Example-1

A vibrometer has a period of free vibration of 2 seconds. It is attached to a machine with a vertical harmonic frequency of 1 Hz. If the vibrometer mass has an amplitude of 2.5mm relative to the vibrometer frame what is the amplitude of vibration of machine ?

time period $\tau = 2$ sec $\qquad Z = 2.5$ mm

$\omega = 1 \times 2\pi = 2\pi$ rad/sec.

Natural frequency $\omega_n = \frac{2\pi}{\tau} = \frac{2\pi}{2} = \pi$ rad/se.

$\xi = 0$, for vibrometers.

Now using the relation

$$\frac{Z}{Y} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \cancel{\left(2\xi\frac{\omega}{\omega_n}\right)^2}}}$$

$$\Rightarrow \frac{Z}{Y} = \frac{(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2}}$$

$$\Rightarrow \frac{2.5}{Y} = \frac{2^2}{\sqrt{(1-2^2)^2}}$$

$$\Rightarrow Y = \frac{2.5 \times \sqrt{(1-2^2)^2}}{2^2} = \boxed{1.875 \text{ mm}}$$

which is the amplitude of vibration of support of m/c in this case.

Example- 2

A seismic instrument having natural frequency of 5Hz is used to measure the vibration of a machine operating at 110 rpm. The relative displacement of seismic mass as read from the instrument is 0.02m. Determine the amplitude of vibration of the machine. Neglect damping.

Data given:-

$f_n = 5$Hz, $N = 110$ rpm, $Z = 0.02$ m $\quad \xi = 0$.

Now $\omega_n = 2\pi f_n = 10\pi$ rad/sec. , $\omega = \frac{2\pi N}{60} = 11.52$ rad/s.

For a vibrometer, the governing equation is

$$\frac{Z}{Y} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

Neglecting damping, we have

$$\frac{Z}{Y} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2}}$$

$$\Rightarrow \frac{0.02}{Y} = \frac{\left(\frac{11.52}{31.416}\right)^2}{\left[1-\left(\frac{11.52}{31.416}\right)^2\right]}$$

$$\Rightarrow \boxed{Y = 0.129\ m}$$

Example - 3

A commericial vibrometer having amplitude of vibration of the m/c part as 5mm and damping factor $\xi = 0.2$, performs harmonic motion. If the difference between the max^m and minimum recorded value is 12 mm and the frequency of vibrating part is 15 rad/sec, find out the natural frequency of vibrometer.

Given data! -

$Y = 5\,mm \quad \xi = 0.2 \quad Z = \frac{12}{2} = 6\,mm \quad \omega = 15\ rad/sec$

Using the relation

$$\frac{Z}{Y} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$\Rightarrow \left(\frac{6}{5}\right)^2 = \frac{(\omega/\omega_n)^4}{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\times0.2\times\frac{\omega}{\omega_n}\right)^2}$$

$$\Rightarrow 1.44 = \frac{(\omega/\omega_n)^4}{(1-\omega^2/\omega_n^2)^2 + (0.4\,\omega/\omega_n)^2}$$

$$\Rightarrow 1.44 + 1.44\frac{\omega^4}{\omega_n^4} - 1.84\frac{\omega^2}{\omega_n^2} = \frac{\omega^4}{\omega_n^4}$$

$$\Rightarrow 0.44\frac{\omega^4}{\omega_n^4} - 1.84\frac{\omega^2}{\omega_n^2} + 1.44 = 0$$

Solving the above equation we have $\frac{\omega}{\omega_n} = 1.772$

$$\Rightarrow \omega_n = \frac{\omega}{1.772} = \frac{15}{1.772} = \boxed{8.465\ rad/sec}.$$

So $f_n = \frac{\omega_n}{2\pi} = \frac{8.465}{2\pi} = \boxed{1.35\ Hz.}$

Example-4

A vibrometer indicates 1% error in measurement and its natural frequency is 4 Hz. If the lowest frequency that can be measured is 36 Hz, find the value of damping factor.

Since the reading recorded by vibrometer is z

So $z = 1.01\ y$

Now $\frac{z}{y} = \frac{(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\xi \frac{\omega}{\omega_n})^2}}$

$\Rightarrow 1.01 = \frac{(36/4)^2}{\sqrt{\{1-(36/4)^2\}^2 + (2\xi \cdot 36/4)^2}}$

$\Rightarrow 1.01 = \frac{81}{\sqrt{6400 + 324\xi^2}}$

$\Rightarrow (1.01)^2 = \frac{81^2}{6400 + 324\xi^2}$

$\Rightarrow \boxed{\xi = 0.313}$

~~Velocity Pick-ups (Vibrometer):-~~

~~Vibrometer is used to measure the displacement of a vibrating body.~~

~~Considering the equation~~

$$\cancel{\frac{z}{y} = \frac{(\omega/\omega_n)^2}{\sqrt{\{1-(\frac{\omega}{\omega_n})^2\}^2 + (2\xi \frac{\omega}{\omega_n})^2}}}$$

Velocity Pick-ups / velometers :-

velocity of the vibrating system can be expressed from the equation

$$y = Y \sin\omega t \quad \text{——(1)}$$

so velocity $\dot{y} = \omega Y \cos\omega t$ ——(2)

Now we have the equation

$$\frac{Z}{Y} = \frac{(\omega/\omega_n)^2}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \quad \text{——(3)}$$

$$\Rightarrow Z = \frac{Y.(\omega/\omega_n)^2}{\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

The relative velocity

$$\dot{z} = \frac{\omega . Y \left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \cos(\omega t - \phi) \quad \text{——(4)}$$

for $\omega/\omega_n \gtrsim 1$

$$\frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \approx 1 \quad \text{——(5)}$$

so from eq. (4)

$$\boxed{\dot{z} = \omega Y \cos(\omega t - \phi)} \quad \text{——(6)}$$

comparing eq. (2) and (6) it can be obser bed that for a phase difference ϕ $\dot{z}$ gives velocity of base as long as eq. (5) is satisfied and this is possible for large value of $\left(\frac{\omega}{\omega_n}\right)$, else velocity of system can be computed from eq. (4).

Acceleration measuring instruments or Accelerometer

Accelerometer is used to measure the acceleration of a vibrating body.

If $\left(\frac{\omega}{\omega_n}\right) \leq 1$, the equation for relative amplitude reduces to

$$\frac{Z}{Y} = \left(\frac{\omega}{\omega_n}\right)^2$$

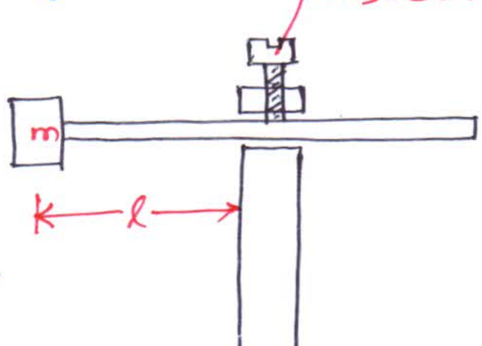
or $$\boxed{Z = \frac{\omega^2 Y}{\omega_n^2} = \frac{\text{Acceleration}}{\omega_n^2}} \quad \text{--- (1)}$$

The expression $\omega^2 Y$ in the above equation is equal to the acceleration amplitude of the body vibrating with frequency ω and having a displacement amplitude Y. So the amplitude recorded Z, under these conditions is proportional to the acceleration of the vibrating body, as ω_n is a constant for the instrument.

Frequency Measuring Instruments:-

The frequency measuring devices are based on resonance principle. For the frequency less than about 100Hz. Two types of reed tachometers are quite useful. Two types of reed tachometers are generally used.

(a) Single-Reed Instrument:-

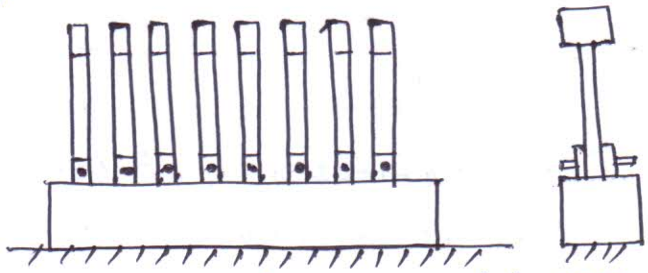


The instrument consists of a cantilever strip, held in a clamp at one end while a mass is attached at the other end. The free length of the strip can be adjusted by means of a screw mechanism. Since each length of strip correspond to a different natural frequency, so the value of natural frequency are marked along length of the reed. The instrument is held firmly against the vibrating member and free length of the strip is altered untill at one particular length, resonance occur. The frequency is then directly read from the strip.

- The instrument is also known as Fullarton Tachometer

(b) Multi Reed Instruments:-

The instrument is also called Frahm Tachometer. It essentially consists of a series of cantilevered reeds carrying small



concentrated mass at their tips. Each reed has a different natural frequency so it is possible to cover a wide frequency range. In practise the instrument is mounted on the vibrating body. The reed whose natural frequency matches with the unknown frequency of the body will undergo resonance and vibrate with larger amplitude. The frequency of the vibrating body can be then found from the known natural frequency of that reed

MODULE 2

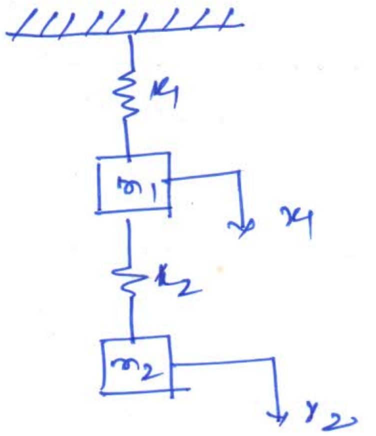
MODULE - II

TWO DEGREE OF FREEDOM SYSTEMS :-

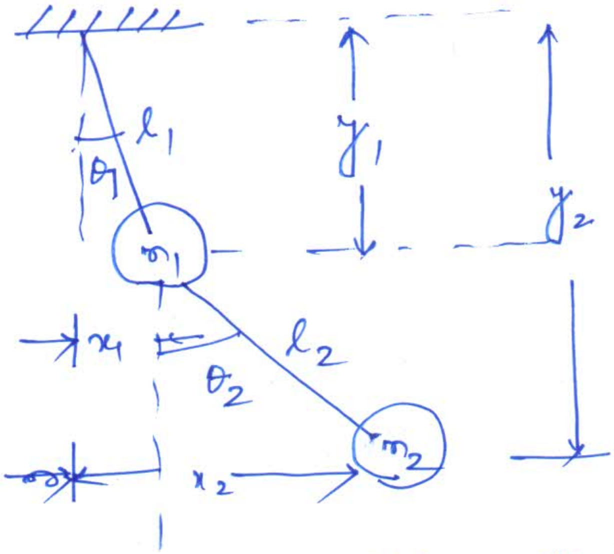
In the preceding sections, systems having single dof have been discussed. In this case the systems have one natural frequency and require only one independent coordinate to describe the system completely. Systems having two dof are important and they introduce the coupling phenomenon where the motion of any of the two independent coordinates, depend also on the other coordinate through spring coupling or dashpots.

- These systems require two independent coordinates to describe their motion.

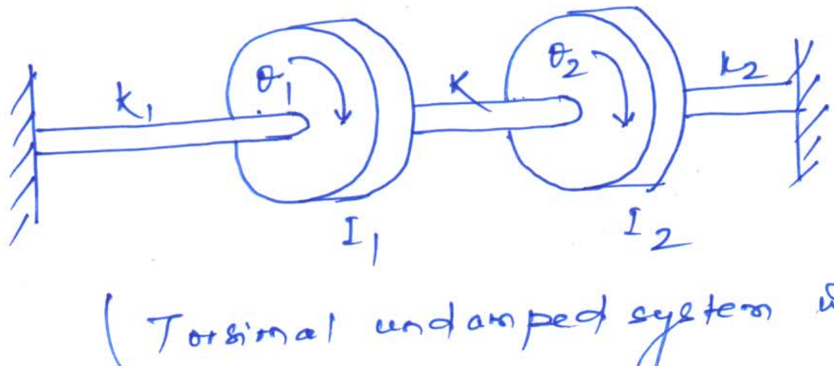
Examples :-



Spring mass ~~damp~~ system

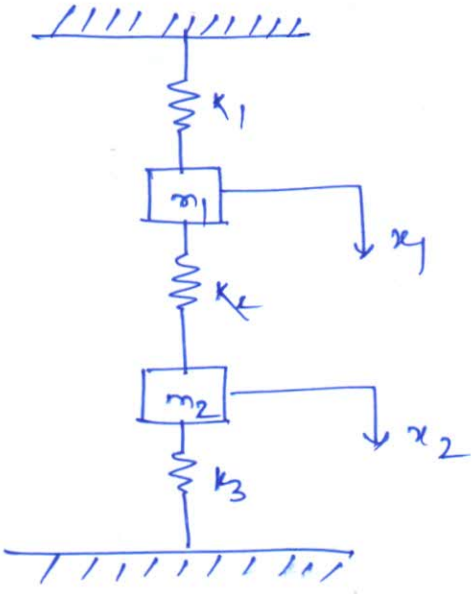


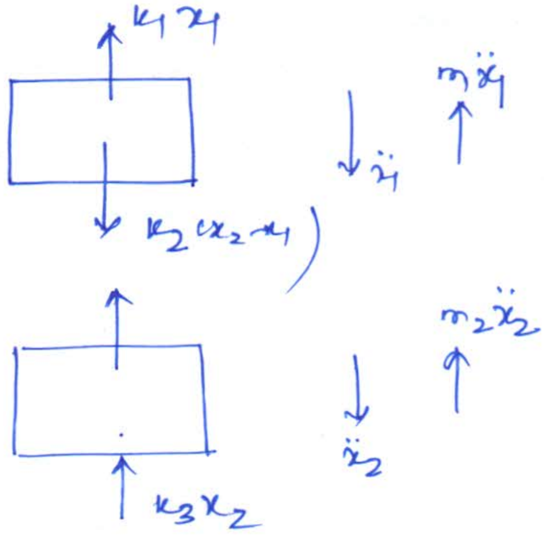
(Double Pendulum system)



(Torsional undamped system with two mass)

Principal mode of vibration :-





Considering an ideal case of two dof system (spring mass system)

Let $x_1, x_2 \rightarrow$ displacements of mass m_1 and m_2 at any instance measured from equilibrium position respectively

Assuming $x_2 > x_1$

The differential equation of motion for the system may be expressed as:

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1) - k_3 x_2 \end{aligned}\right\} \text{---}(1)$$

or

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) &= 0 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3 x_2 &= 0 \end{aligned}\right\} \text{---}(2)$$

or

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 &= k_2 x_2 \\ m_2\ddot{x}_2 + (k_2 + k_3)x_2 &= k_2 x_1 \end{aligned}\right\} \text{---}(3)$$

Now assuming a solution for x_1 and x_2 under steady state conditions

$$\left.\begin{aligned} x_1 &= X_1 \sin\omega t \\ x_2 &= X_2 \sin\omega t \end{aligned}\right\} \text{---}(4)$$

Where X_1 and X_2 are the amplitude of two masses and ω is the frequency of harmonic motion.

From eq. (4)

$$x_1 = X_1 \sin\omega t \qquad x_2 = X_2 \sin\omega t$$
$$\dot{x}_1 = \omega X_1 \cos\omega t \qquad \dot{x}_2 = \omega X_2 \cos\omega t$$
$$\ddot{x}_1 = -\omega^2 X_1 \sin\omega t \qquad \ddot{x}_2 = -\omega^2 X_2 \sin\omega t \quad \text{--- (5)}$$

Substituting the values of eq. (5) in eq. (3) and cancelling common term $\sin\omega t$ at later stage

$$\left.\begin{aligned} -m_1\omega^2 X_1 \sin\omega t + (k_1+k_2) X_1 \sin\omega t &= k_2 X_2 \sin\omega t \\ -m_2\omega^2 X_2 \sin\omega t + (k_2+k_3) X_2 \sin\omega t &= k_2 X_1 \sin\omega t \end{aligned}\right\} \text{--- (6)}$$

or
$$\left.\begin{aligned} \{-m_1\omega^2 + (k_1+k_2)\} X_1 &= k_2 X_2 \\ \{-m_2\omega^2 + (k_2+k_3)\} X_2 &= k_2 X_1 \end{aligned}\right\} \text{--- (7)}$$

Eq. (7) gives two equations

$$\frac{X_1}{X_2} = \frac{k_2}{\{(k_1+k_2) - m_1\omega^2\}} \quad \text{--- (8)}$$

$$\frac{X_1}{X_2} = \frac{(k_2+k_3) - m_2\omega^2}{k_2} \quad \text{--- (9)}$$

Equating Eq. (8) and (9)

$$\frac{k_2}{[(k_1+k_2) - m_1\omega^2]} = \frac{[(k_2+k_3) - m_2\omega^2]}{k_2}$$

$$\Rightarrow k_2^2 = [(k_1+k_2) - m_1\omega^2][(k_2+k_3) - m_2\omega^2]$$

$$\Rightarrow m_1 m_2\omega^4 - [m_1(k_2+k_3) + m_2(k_1+k_2)]\omega^2 + [k_1k_2 + k_1k_3 + k_2k_3] = 0 \quad \text{--- (10)}$$

Eq. (10) gives two values of ω^2 and therefore two positive value of ω corresponding to the two natural frequencies ω_{n_1} and ω_{n_2} of the system. Eq. (10) is called frequency equation as the roots of this equation gives the natural frequencies

~~of the system.~~

~~Now let $m_1 = m_2 = m$ and $k_1 = k_3 = k$ } — (11)~~

~~Eq. (10) reduces to~~

$$\cancel{m^2\omega^4 - 2m(k+k_2)\omega^2 + (k^2+2kk_2)\,0}$$

~~which gives~~

$$\cancel{\omega_{n_1}, \omega_{n_2} = \sqrt{\frac{(k+k_2) \pm k_2}{m}}}$$

$$\Rightarrow \quad \omega^4 - \left[\frac{k_1+k_3}{m_2} + \frac{k_1+k_2}{m_1}\right]\omega^2 + \frac{k_1k_2 + k_1k_3 + k_2k_3}{m_1 m_2} = 0 \quad \text{— (10)}$$

Equation (10) gives two values of ω^2 and therefore two positive values of ω corresponding to the two natural frequencies ω_{n_1} and ω_{n_2} of the system. Eq. (10) is called the frequency equation as roots of this equation gives the natural frequency of the system.

Now let $m_1 = m_2 = m$ and $k_1 = k_3 = k$ } — (11)

So equation (10) reduces to

$$\cancel{\omega^4 - \left[\frac{2K}{m} + \frac{k_1+k_2}{m}\right]\omega^2 + \frac{kk_2 + k^2 + k_2k}{m^2} = 0}$$

$$\Rightarrow \cancel{\omega^4 - \left(\frac{2K}{m} + \frac{k_1+k_2}{m}\right)\omega^2 + \frac{2kk_2 + k^2}{m^2} = 0}$$

$$\cancel{\omega^2 = 2K+}$$

$$m^2\omega^4 - 2m(k+k_2)\omega^2 + (k^2 + 2kk_2) = 0$$

which gives

$$\omega_{n_1}, \omega_{n_2} = \sqrt{\frac{(k+k_2) \pm k_2}{m}}$$

or

$$\left.\begin{aligned} \omega_{n_1} &= \sqrt{\frac{k}{m}} \\ \omega_{n_2} &= \sqrt{\frac{k+2k_2}{m}} \end{aligned}\right\} \quad \text{— (12)}$$

substituting the condition of Eq. (11), in Eq. (8) and Eq. (9) can be reduced to

$$\frac{X_1}{X_2} = \frac{K_2}{[(K_1+K_2) - m\omega^2]} \quad \text{--- (13)}$$

$$\frac{X_1}{X_2} = \frac{[(K_2+K) - m\omega^2]}{K_2} \quad \text{--- (14)}$$

Now substituting the value of ω_{n_1} in eq. (12) in any of the Eq. (13) and Eq. (14), we have

$$\boxed{\frac{X_1}{X_2} = +1}$$

It means the system is vibrating with the first natural frequency ω_{n_1}, the mode shape is such that the ratio of amplitude is +1

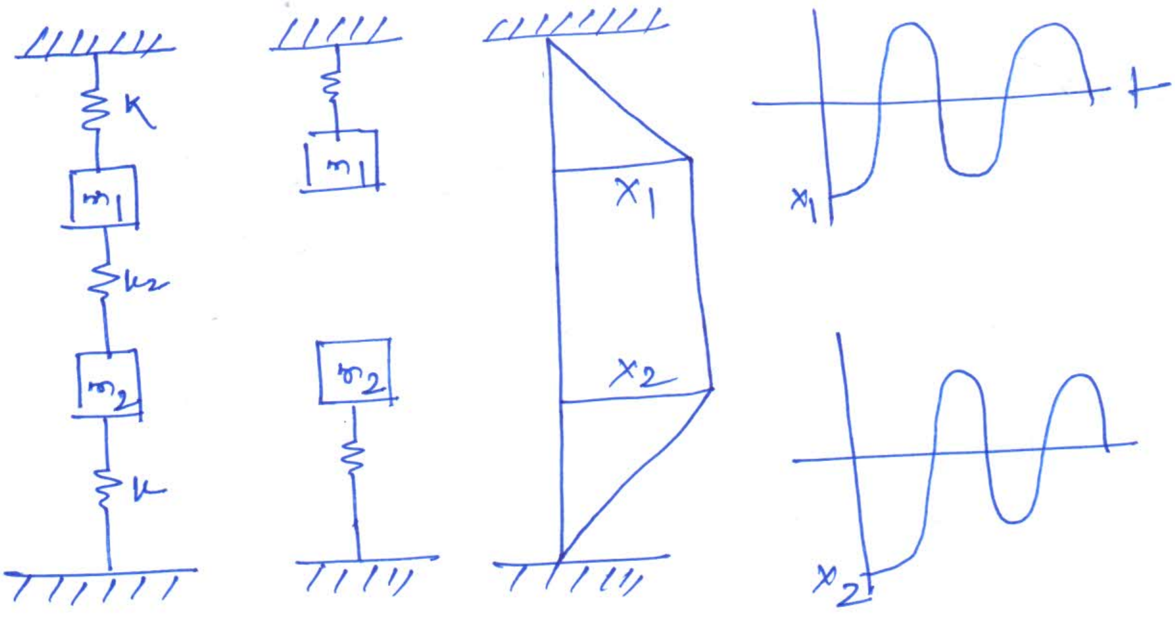
So $\left(\frac{X_1}{X_2}\right)_1 \rightarrow$ Ratio of amplitude in the first modeshape corresponding to first natural frequency ω_{n_1}

Now substituting the value of ω_{n_2} from eq. (12) in eq. (13) or eq. (14), we have

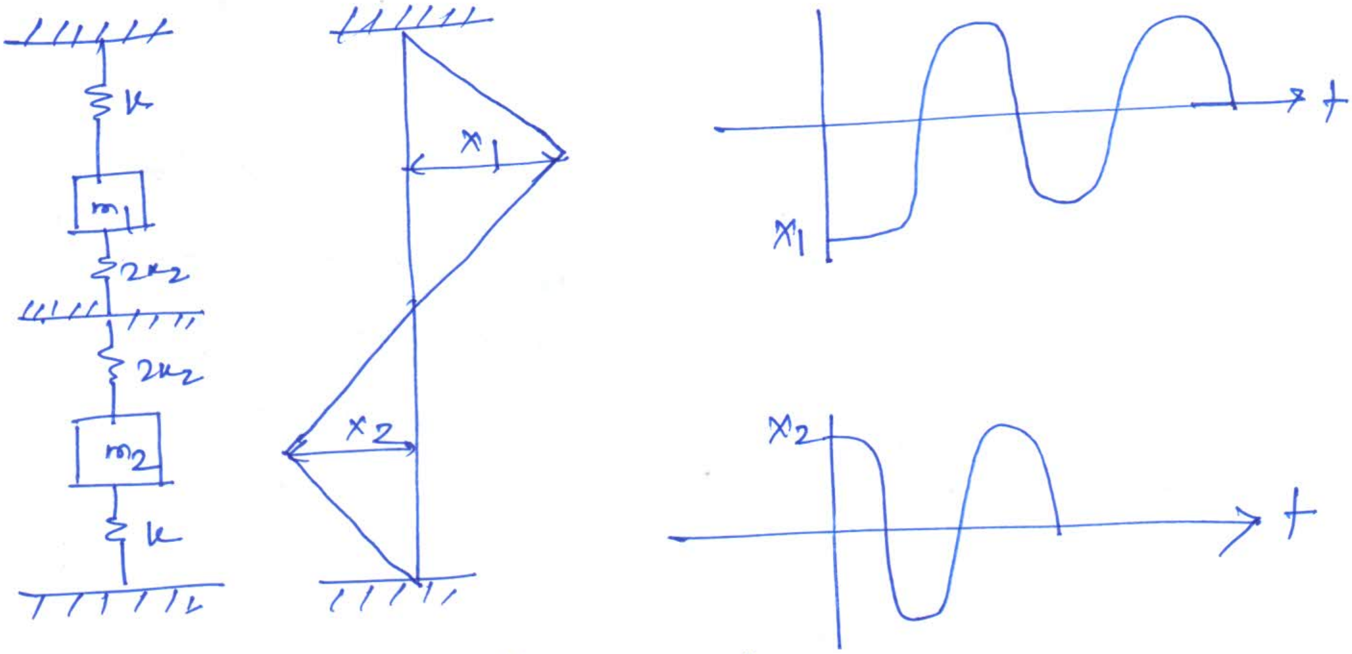
$$\boxed{\left(\frac{X_1}{X_2}\right) = -1}$$

and $\left(\frac{X_1}{X_2}\right)_2 \rightarrow$ indicates second mode shape corresponding to second natural frequency ω_{n_2}

- The ratio of amplitudes of two mass being +1, indicates the amplitudes are equal and two motions are in phase i.e the two masses move (up and down together.
- The ratio of amplitude of two masses being −1 means the amplitudes are equal but the motions are out of phase i.e when the mass moving down the other mass is moving up and vice versa.



(1st mode)



(2nd Mode)

It can be seen that if the two masses are given equal initial displacement in the same direction and released they will vibrate in 1st principal mode of vibration with first natural frequency. Also if they are given equal initial displacement in opposite direction, and released they will vibrate in second principal mode of vibration with second natural frequency.

- However, if the the two masses are given unequal initial displacement in any direction their motion will be the super position of two harmonic motions corresponding to the two natural frequencies as:

$$\left.\begin{aligned} x_1 &= x_1' \cos \omega_{n_1} t + x_1'' \cos \omega_{n_1} t \\ x_2 &= x_2' \cos \omega_{n_2} t + x_2'' \cos \omega_{n_2} t \end{aligned}\right\} \quad (15)$$

where x_1' and x_1'' $\rightarrow$ amplitudes of mass m_1 at lower and higher frequencies respectively

x_2' and x_2'' $\rightarrow$ amplitude of mass m_2 at lower and higher natural frequencies.

and they will have the relationship

$$\left.\begin{aligned} \left(\frac{x_1'}{x_2'}\right) &= \left(\frac{x_1}{x_2}\right)_1 \\ \left(\frac{x_1''}{x_2''}\right) &= \left(\frac{x_1}{x_2}\right)_2 \\ x_1' + x_1'' &= \text{initial displacement of } m_1 \\ x_2' + x_2'' &= \text{initial displacement of } m_2 \end{aligned}\right\} \quad (16)$$

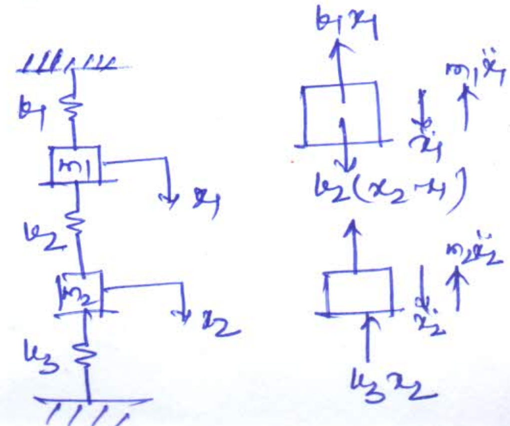
Example

For the system shown in the figure find two natural frequencies when:

$m_1 = m_2 = m = 9.8$ kg.

$k_1 = k_3 = 8820$ N/m

$k_2 = 3430$ N/m.



find out the resultant motions of m_1 and m_2 for the following different cases:

(a) both masses are displaced 5mm in downward directi and released simultaneously

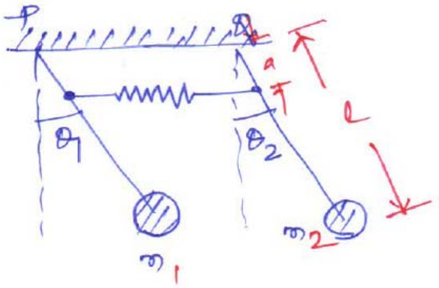
(b) both masses are displaced 5mm; m_1 in downward direction and m_2 in upward direction and released simultaneously

(c) mass m_1 is displaced 5mm downward and mass m_2 is displaced 7.5mm downward and released simultaneously

(d) mass m_1 displaced 5mm upward while m_2 is fixed and both masses are released simultaneously.

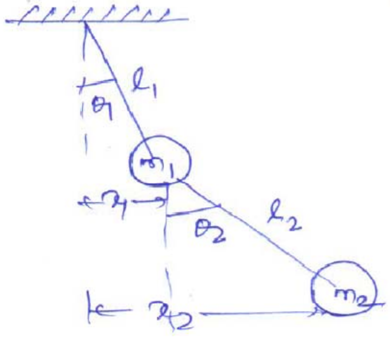
Assignment

1. Determine the normal modes of vibrations of the coupled pendulum as shown in the figure.

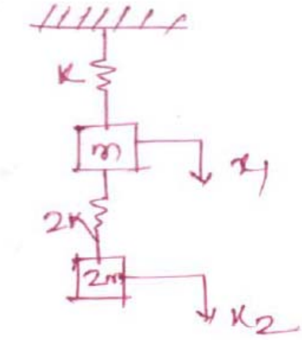


~~The equation of motion are,~~
Derive the equation of motion of the two masses and find natural frequencies of the system when
$k = 150\ N/m$
$m_1 = 3\ kg$, $m_2 = 5\ kg$.
$l = 0.3\ m$ $a = 0.15m$.

Q.2 Setup the differential equations of motion for the double pendulum shown in the figure using coordinates x_1 and x_2 and assuming small amplitudes. Find the natural frequencies, ratios of amplitude and draw the mode shapes if $m_1 = m_2 = m$ and $l_1 = l_2 = l$

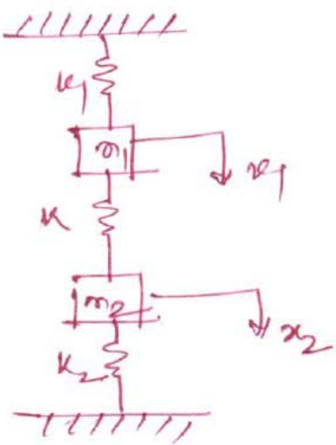


Q



Determine the natural frequencies and amplitude of ratios of the system. Determine the response of the system at $K = 1000\,N/m$ and $m = 20\,Kg$.

Q.2



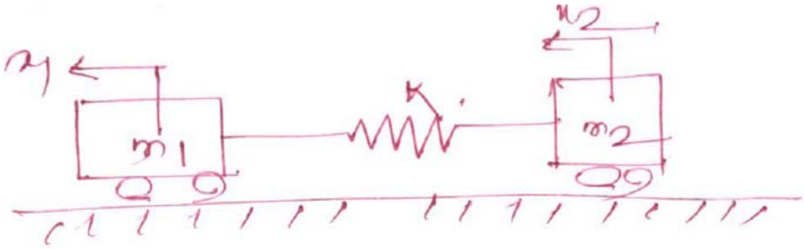
$k_1 = k_2 = 40\,N/m$

$K = 60\,N/m$,

$m_1 = m_2 = 10\,Kg$,

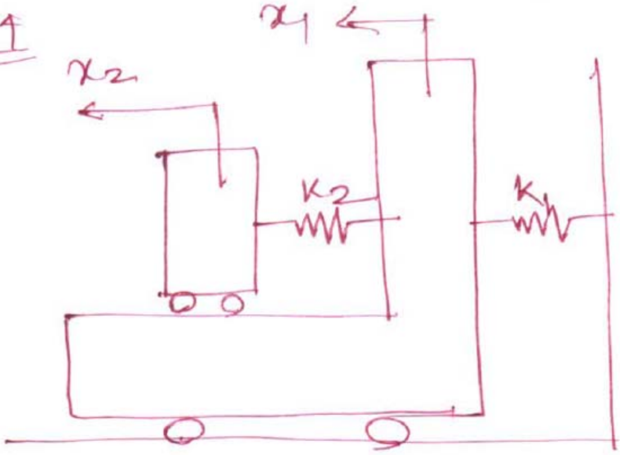
Determine the natural freq. of the system.

Q.3



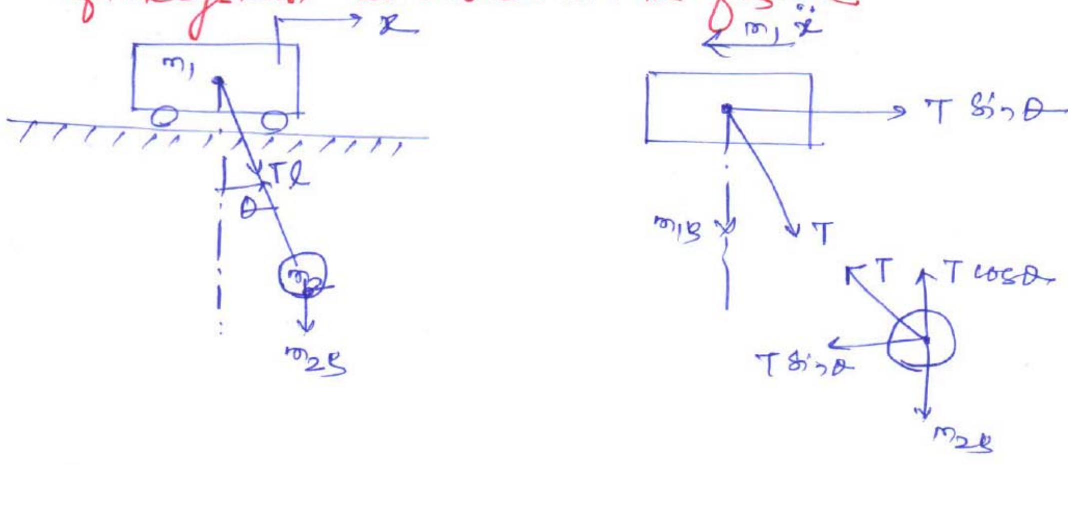
$m_1 = 20$ Kg, $m_2 = 35$ Kg, $K = 300$ N/m.
Determine natural frequency.

Q.4



$m_1 = 200$ Kg $m_2 = 50$ Kg.
$k_1 = 100,000$ N/m
$k_2 = 200,000$ N/m.

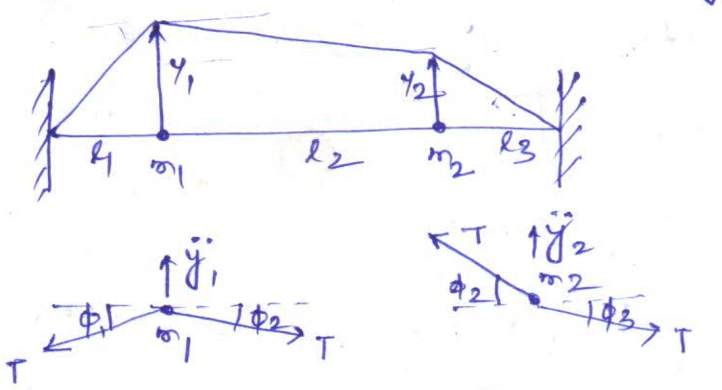
Q.3 Find the natural frequencies of vibration of the system as shown in the figure.



Other cases of simple two dof systems:-

Different two dof systems are discussed in this section to find out the natural frequencies and corresponding mode shapes.

1. Two masses fixed on a tightly stretched string:-



Consider two masses fixed on a tight string stretched between two supports and having tension T.

Let the amplitude of vibration is small and tension T is large.

At any instant let y_1 and y_2 be the displacement of two masses respectively.

The equation of lateral motion of the masses are:

$$\left.\begin{aligned} m_1\ddot{y}_1 + T\sin\phi_1 + T\sin\phi_2 = 0 \\ m_2\ddot{y}_2 - T\sin\phi_2 + T\sin\phi_3 = 0 \end{aligned}\right\} \quad -(1)$$

Now we have

$$\left.\begin{aligned} \sin\phi_1 &= \frac{y_1}{l_1} \\ \sin\phi_2 &= \frac{y_1 - y_2}{l_2} \\ \sin\phi_3 &= \frac{y_2}{l_3} \end{aligned}\right\} \quad -(2)$$

Substituting the values of eq. (2) in eq. (1)

$$m_1\ddot{y}_1 + T\frac{y_1}{l_1} + T\left(\frac{y_1 - y_2}{l_2}\right) = 0$$

$$m_2\ddot{y}_2 - T\left(\frac{y_1 - y_2}{l_2}\right) + T\frac{y_2}{l_3} = 0$$

or

$$\left.\begin{aligned} m_1\ddot{y}_1 + \left(\frac{T}{l_1} + \frac{T}{l_2}\right)y_1 = \frac{T}{l_2}y_2 \\ m_2\ddot{y}_2 + \left(\frac{T}{l_2} + \frac{T}{l_3}\right)y_2 = \frac{T}{l_2}y_1 \end{aligned}\right\} \quad -(3)$$

Assuming a steady state solution for principal mode vibration

$$\left.\begin{aligned} y_1 &= Y_1 \sin \omega t \\ y_2 &= Y_2 \sin \omega t \end{aligned}\right\} \quad -(4)$$

~~subs~~ from eq.(4) we have

$$\dot{y}_1 = \omega Y_1 \cos \omega t \qquad \dot{y}_2 = \omega Y_2 \cos \omega t$$

$$\ddot{y}_1 = -\omega^2 Y_1 \sin \omega t \qquad \ddot{y}_2 = -\omega^2 Y_2 \sin \omega t$$

substituting the values in eq.(3)

$$-m_1 \omega^2 Y_1 \sin \omega t + \left(\frac{T}{l_1} + \frac{T}{l_2}\right) Y_1 \sin \omega t = \frac{T}{l_2} Y_2 \sin \omega t$$

$$-m_2 \omega^2 Y_2 \sin \omega t + \left(\frac{T}{l_2} + \frac{T}{l_3}\right) Y_2 \sin \omega t = \frac{T}{l_2} Y_1 \sin \omega t$$

or

$$\left.\begin{aligned} \left[-m_1 \omega^2 + \left(\frac{T}{l_1} + \frac{T}{l_2}\right)\right] Y_1 &= \frac{T}{l_2} Y_2 \\ \left[-m_2 \omega^2 + \left(\frac{T}{l_2} + \frac{T}{l_3}\right)\right] Y_2 &= \frac{T}{l_2} Y_1 \end{aligned}\right\} \quad -(5)$$

from eq.(5) the ratio of amplitude of vibration can be obtained as

$$\frac{Y_1}{Y_2} = \frac{T/l_2}{\left[\left(\frac{T}{l_1} + \frac{T}{l_2}\right) - m_1 \omega^2\right]} \quad -(6)$$

$$\frac{Y_1}{Y_2} = \frac{\left[\left(\frac{T}{l_2} + \frac{T}{l_3}\right) - m_2 \omega^2\right]}{T/l_2} \quad -(7)$$

frequency equation can be obtained by equating eq.(6) and (7)

$$\frac{T/l_2}{\left(\frac{T}{l_1} + \frac{T}{l_2}\right) - m_1 \omega^2} = \frac{\left(\frac{T}{l_2} + \frac{T}{l_3}\right) - m_2 \omega^2}{T/l_2}$$

$$\left[\left(\frac{T}{l_1}+\frac{T}{l_2}\right)-m_1\omega^2\right]\left[\left(\frac{T}{l_2}+\frac{T}{l_3}\right)-m_2\omega^2\right]=\frac{T^2}{l_2^2}$$

$$\Rightarrow \quad m_1m_2\omega^4-\left[m_1\left(\frac{T}{l_2}+\frac{T}{l_3}\right)+m_2\left(\frac{T}{l_1}+\frac{T}{l_2}\right)\right]\omega^2 + \frac{T^2}{l_1l_2}+\frac{T^2}{l_1l_3}+\frac{T^2}{l_2l_3}=0 \quad \text{---(8)}$$

Assuming

$$\left.\begin{array}{l} m_1=m_2=m \\ l_1=l_2=l_3=l \end{array}\right\} \text{---(9)}$$

We have

$$m^2\omega^4-\left[m\left(\frac{2T}{l}\right)+m\left(\frac{2T}{l}\right)\right]\omega^2+\frac{3T^2}{l^2}=0$$

or

$$m^2\omega^4-\frac{4mT}{l}\omega^2+\frac{3T^2}{l^2}=0 \quad \text{---(10)}$$

solving for ω, the two values of natural frequencies are

$$\left.\begin{array}{l} \omega_{n_1}=\sqrt{\dfrac{T}{ml}} \\ \omega_{n_2}=\sqrt{\dfrac{3T}{ml}} \end{array}\right\} \text{---(11)}$$

The ratio of vibration amplitude can be expressed as.

$$\left.\begin{array}{l} \dfrac{y_1}{y_2}=\dfrac{T/l}{\frac{2T}{l}-m\omega^2} \\[2ex] \dfrac{y_1}{y_2}=\dfrac{\frac{2T}{l}-m\omega^2}{T/l} \end{array}\right\} \text{---(12)}$$

or

$$\frac{T/l}{\frac{2T}{l}-m\omega^2}=\frac{2T/l-m\omega^2}{T/l}$$

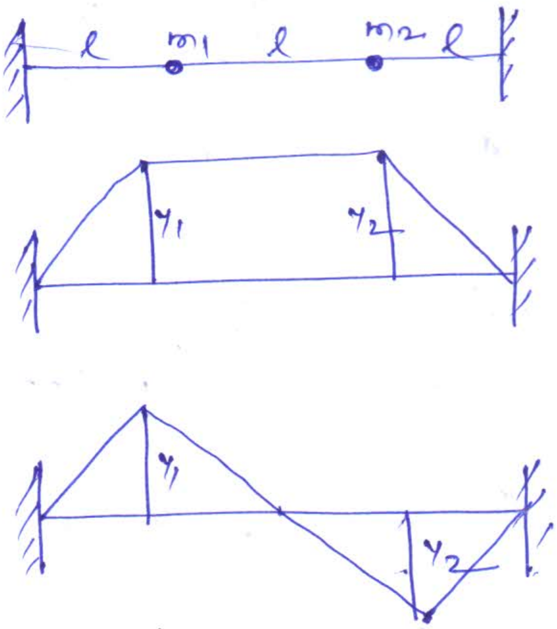
$$\Rightarrow \quad \frac{2T}{l}-m\omega^2=\mp\frac{T}{l}$$

$$\left.\begin{array}{l} \omega_{n_1}=\sqrt{\dfrac{T}{ml}} \\ \omega_{n_2}=\sqrt{\dfrac{3T}{ml}} \end{array}\right\} \text{---(11)}$$

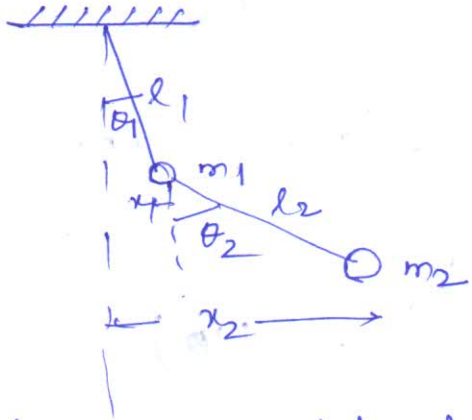
The corresponding principal mode shapes are obtained by substituting in either of the equation (10) the value ω_{n_1} and ω_{n_2}

$$\left(\frac{y_1}{y_2}\right)_1 = +1$$

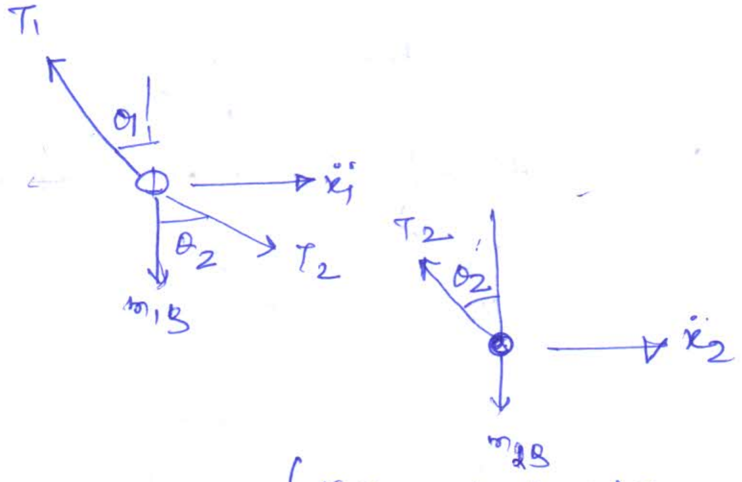
$$\left(\frac{y_1}{y_2}\right)_2 = -1$$



Double Pendulum :-



(Double pendulum)



(FBD of double pendulum)

Let m_1, m_2 = masses of two ~~pend~~ balls respectively

l_1, l_2 = length of strings,

from the geometry

$$\left.\begin{aligned} \sin\theta_1 &= \frac{x_1}{l_1} \\ \sin\theta_2 &= \frac{x_2 - x_1}{l_2} \end{aligned}\right\} \quad -(1)$$

Considering no vertical motion and resolving the vertical components,

$$\left.\begin{aligned} T_2\cos\theta_2 &= m_2 g \\ T_1\cos\theta_1 &= m_1 g + T_2\cos\theta_2 \end{aligned}\right\} \quad (2)$$

For small values of θ_1 and θ_2

$$\left.\begin{aligned} T_2 &= m_2 g \\ T_1 &= m_1 g + T_2 = (m_1+m_2) g \end{aligned}\right\} \quad (3)$$

Now the differential equation of motion of the two masses in horizontal direction

$$\left.\begin{aligned} m_1\ddot{x}_1 + T_1\sin\theta_1 - T_2\sin\theta_2 &= 0 \\ m_2\ddot{x}_2 + T_2\sin\theta_2 &= 0 \end{aligned}\right\} \quad (4)$$

substituting the values of T_1 and T_2 and $\sin\theta_1$ and θ_2 in above equation we have

$$m_1\ddot{x}_1 + (m_1+m_2) g \cdot \frac{x_1}{l_1} - m_2 g\left(\frac{x_2-x_1}{l_2}\right) = 0$$

$$m_2\ddot{x}_2 + m_2 g\left(\frac{x_2-x_1}{l_2}\right) = 0$$

or

$$\left.\begin{aligned} m_1\ddot{x}_1 + \left[\frac{(m_1+m_2)}{l_1} + \frac{m_2}{l_2}\right] x_1 g &= \frac{m_2}{l_2} \cdot x_2 g \\ m_2\ddot{x}_2 + \frac{m_2}{l_2} g x_2 &= \frac{m_2}{l_2} g x_1 \end{aligned}\right\} \quad (5)$$

Assuming a steady solution for the principal mode of vibration

$$\left.\begin{aligned} x_1 &= X_1 \sin\omega t \\ x_2 &= X_2 \sin\omega t \end{aligned}\right\} \quad (6)$$

From equation (6)

$$\dot{x}_1 = \omega X_1 \cos\omega t \qquad \dot{x}_2 = \omega X_2 \cos\omega t$$

$$\ddot{x}_1 = -\omega^2 X_1 \sin\omega t \qquad \ddot{x}_2 = -\omega^2 X_2 \sin\omega t \quad (7)$$

substituting the values of $x_1, \dot{x}_1, \ddot{x}_1$ and $x_2, \dot{x}_2, \ddot{x}_2$ in equation (5), we have

$$-m_1\omega^2 x_1 \sin\omega t + \left[\frac{m_1+m_2}{l_1} + \frac{m_2}{l_2}\right] g \cdot x_1 \sin\omega t = \frac{m_2}{l_2} g \cdot x_2$$

$$-m_2\omega^2 x_2 \sin\omega t + \frac{m_2}{l_2} g\, x_2 \sin\omega t = \frac{m_2}{l_2} g\, x_1 \sin\omega t \quad \text{--- (8)}$$

cancelling out the common term of $\sin\omega t$ from the equation

$$\left.\begin{aligned} \left[-m_1\omega^2 + \left[\frac{m_1+m_2}{l_1} + \frac{m_2}{l_2}\right] g\right] x_1 &= \frac{m_2}{l_2} g\, x_2 \\ \left[-m_2\omega^2 + \frac{m_2}{l_2} g\right] x_2 &= \frac{m_2}{l_2} g\, x_1 \end{aligned}\right\} \text{--- (9)}$$

From equation (9) we have two values of $\frac{x_1}{x_2}$ as:

$$\boxed{\begin{aligned} \frac{x_1}{x_2} &= \frac{m_2/l_2 \cdot g}{\left[\frac{m_1+m_2}{l_1} + \frac{m_2}{l_2}\right] g - m_1\omega^2} \quad \text{--- (10)} \\ \frac{x_1}{x_2} &= \frac{\left(\frac{m_2}{l_2} g - m_2\omega^2\right)}{\frac{m_2}{l_2} g} \quad \text{--- (11)} \end{aligned}}$$

considering a special case of $m_1 = m_2 = m$ and $l_1 = l_2 = l$

Equation (10) and (11) may be written as

$$\frac{x_1}{x_2} = \frac{m/l \cdot g}{\left(\frac{2m}{l} + \frac{m}{l}\right) g - m\omega^2} = \frac{m/l \cdot g}{\frac{3m}{l} g - m\omega^2}$$

$$\Rightarrow \boxed{\frac{x_1}{x_2} = \frac{g/l}{\left(\frac{3g}{l} - \omega^2\right)}} \quad \text{--- (12)}$$

And $$\boxed{\frac{x_1}{x_2} = \frac{\left(\frac{m}{l} g - m\omega^2\right)}{\frac{mg}{l}} = \frac{(g/l - \omega^2)}{(g/l)}} \quad \text{--- (13)}$$

Equating equation (12) and (13)

$$\frac{g/l}{\frac{3g}{l} - \omega^2} = \frac{g/l - \omega^2}{g/l}$$

$$\Rightarrow \frac{3g^2}{l^2} - \frac{3g\omega^2}{l} - \frac{g\omega^2}{l} + \omega^4 = \frac{g^2}{l^2}$$

$$\Rightarrow \boxed{\omega^4 - \frac{4g\omega^2}{l} + \frac{2g^2}{l^2} = 0} \quad \text{--- (14)}$$

or

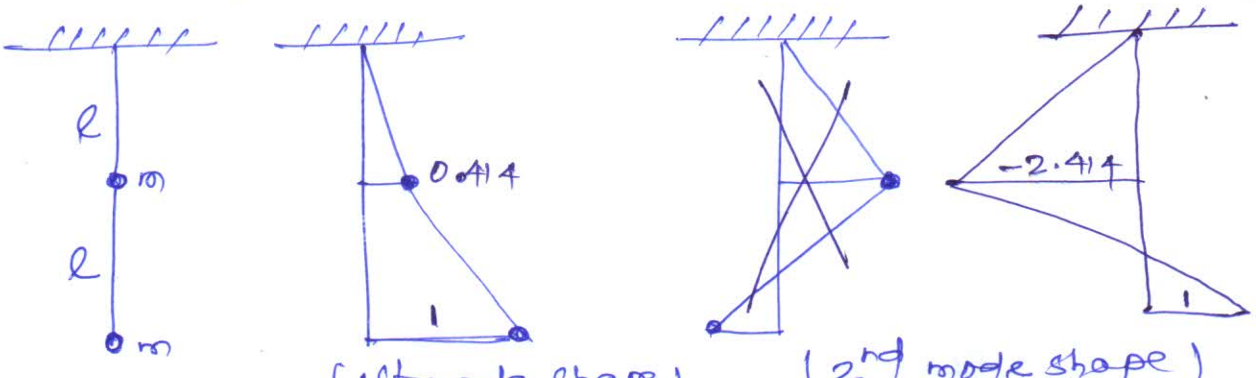
$$\left.\begin{aligned} \omega_{n_1} &= \sqrt{\frac{g}{l}\left(2-\sqrt{2}\right)} \\ \omega_{n_2} &= \sqrt{\frac{g}{l}\left(2+\sqrt{2}\right)} \end{aligned}\right\} \quad \text{--- (15)}$$

corresponding modeshapes can be obtained by substituting the value of ω_{n_1} and ω_{n_2} in equations (12) and (13) for 1st and 2nd mode shapes respectively

So the principal modes are:

$$\boxed{\begin{aligned} \left(\frac{X_1}{X_2}\right)_1 &= \frac{1}{1+\sqrt{2}} = -1+\sqrt{2} = +0.414 \\ \left(\frac{X_1}{X_2}\right)_2 &= \frac{1}{1-\sqrt{2}} = -1-\sqrt{2} = -2.414 \end{aligned}} \quad \text{--- (16)}$$

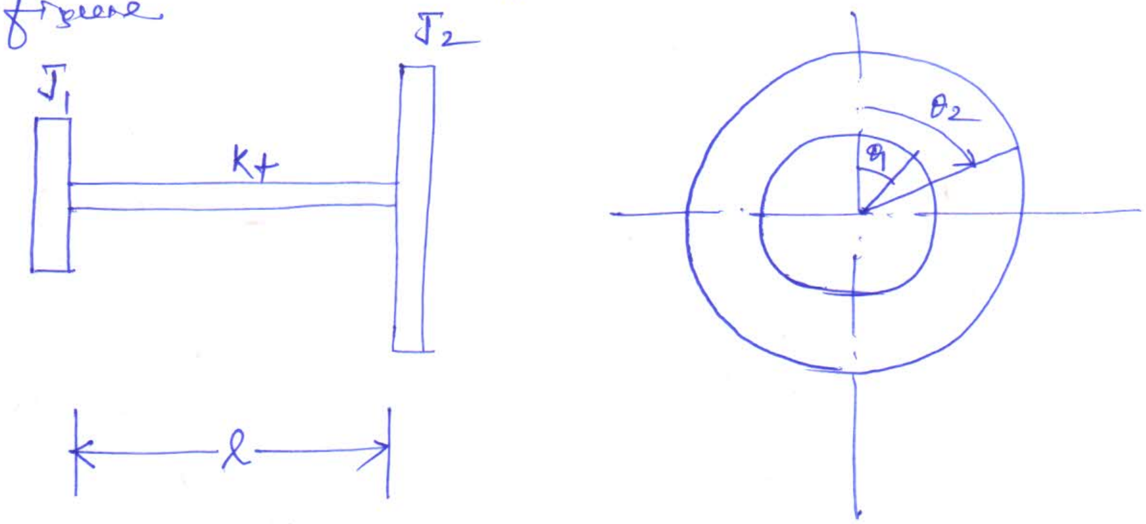
The mode shapes are as shown in the figure:



(1st mode shape) (2nd mode shape)

Torsional system:-

Consider a torsional system with two rotor shown the figure



Let J_1, J_2 = moment of inertia of rotor 1 and rotor 2 respectively

K_t = torsional stiffness of shaft

θ_1, θ_2 = displacement of rotor 1 and 2 respectively at any instant.

θ then twist on the shaft = $\theta_2 - \theta_1$

torque exerted by shaft in the direction of rotation on $J_1 = K_t(\theta_2 - \theta_1)$

and same torque is exerted on J_2 in opposite direction,

The differential equation of motion are:

$$\left.\begin{aligned} J_1 \ddot{\theta}_1 &= K_t(\theta_2 - \theta_1) \\ J_2 \ddot{\theta}_2 &= -K_t(\theta_2 - \theta_1) \end{aligned}\right\} \quad (1)$$

or

$$\left.\begin{aligned} J_1 \ddot{\theta}_1 + K_t\theta_1 &= K_t\theta_2 \\ J_2 \ddot{\theta}_2 + K_t\theta_2 &= K_t\theta_1 \end{aligned}\right\} \quad (2)$$

Assuming the solution for principal mode of vibration as:

$$\left.\begin{aligned} \theta_1 &= B_1 \sin\omega t \\ \theta_2 &= B_2 \sin\omega t \end{aligned}\right\} \quad (3)$$

From equation (3) we have,

$\dot{\theta}_1 = \omega\beta_1 \cos\omega t$ | $\dot{\theta}_2 = \omega\beta_2 \cos\omega t$

$\ddot{\theta}_1 = -\omega^2\beta_1 \sin\omega t$ | $\ddot{\theta}_2 = -\omega^2\beta_2 \sin\omega t$ —(4)

$$\left.\begin{aligned} -J_1\omega^2\beta_1 \sin\omega t + k_t\beta_1 \sin\omega t &= k_t\beta_2 \sin\omega t \\ -J_2\omega^2\beta_2 \sin\omega t + k_t\beta_2 \sin\omega t &= k_t\beta_1 \sin\omega t \end{aligned}\right\} \text{—(5)}$$

or
$$\left.\begin{aligned} (-J_1\omega^2 + k_t)\beta_1 &= k_t\beta_2 \\ (-J_2\omega^2 + k_t)\beta_2 &= k_t\beta_1 \end{aligned}\right\} \text{—(6)}$$

The two ratios obtained from eq. (6) are:

$$\boxed{\begin{aligned} \frac{\beta_1}{\beta_2} &= \frac{k_t}{-J_1\omega^2 + k_t} \quad \text{—(7)} \\ \frac{\beta_1}{\beta_2} &= \frac{-J_2\omega^2 + k_t}{k_t} \quad \text{—(8)} \end{aligned}}$$

Equating the two equations:

$$\frac{k_t}{-J_1\omega^2 + k_t} = \frac{-J_2\omega^2 + k_t}{k_t}$$

$$\Rightarrow J_1J_2\omega^4 - J_1\omega^2 k_t - J_2\omega^2 k_t + k_t^2 = k_t^2$$

$$\Rightarrow \omega^2\left[J_1J_2\omega^2 - (J_1 + J_2)k_t\right] = 0$$

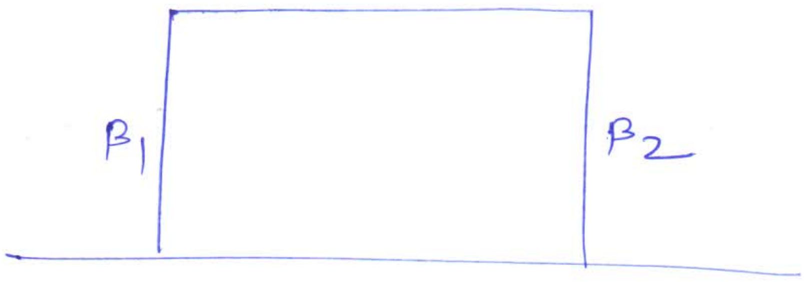
or $$J_1J_2\omega^2 - (J_1 + J_2)k_t = 0$$

or $$\omega_{n_2} = \sqrt{\frac{k_t(J_1 + J_2)}{J_1J_2}} \qquad \omega_{n_1} = 0 \quad \text{—(9)}$$

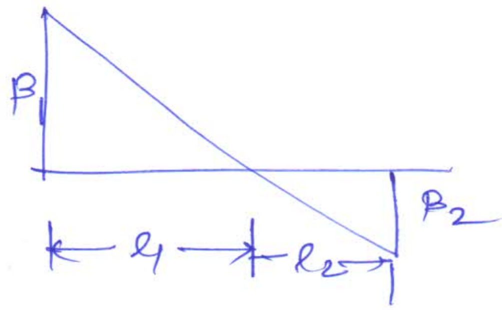
which gives

$$\left(\frac{\beta_1}{\beta_2}\right) = +1 \qquad \left(\frac{\beta_1}{\beta_2}\right) = -\frac{J_2}{J_1} \quad \text{—(10)}$$

The corresponding mode shapes are:



$\omega = 0 = \omega_{n_1}$

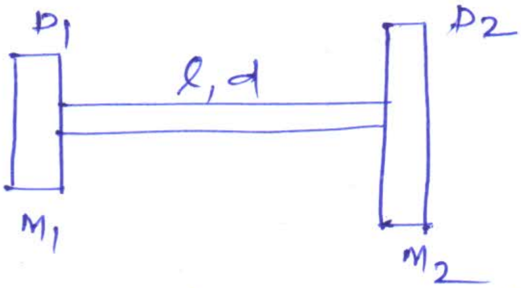


$\omega = \omega_{n_2}$

Example:-

Determine the natural frequency of torsional vibrations of a shaft with two circular discs of uniform thickness at the ends. The masses of the disc are $M_1 = 500$ kg and $M_2 = 1000$ kg and their outer diameter are $D_1 = 125$ cm and $D_2 = 190$ cm. The length of the shaft is $l = 300$ cm and it's diameter $d = 10$ cm. Modulus of rigidity for the material of shaft is $G = 0.83 \times 10^{11}$ N/m^2.

- Also find in what propertion will the natural frequency of this shaft will change if along half the length of the shaft the diameter is increased from 10cm to 20cm.



Given data:- $M_1 = 500$ kg, $M_2 = 1000$ kg,

$D_1 = 1.25$ m $D_2 = 1.9$ m,

length of shaft $l = 300$ cm, $= 3$ m,

dia. of shaft $d = 0.1$ m,

Modulus of rigidity of shaft $G = 0.83 \times 10^{11}$ N/m^2

Now we have

$$J_1 = M_1 \frac{r_1^2}{2} = 500 \times \frac{(1.25/2)^2}{2} = 97.65 \simeq 98\ kg\,m^2$$

$$J_2 = M_2 \cdot \frac{r_2^2}{2} = 1000 \times \frac{(1.9/2)^2}{2} = 451.25\ kg\,m^2$$

$$k_t = \frac{G \cdot I_p}{l} = \frac{0.83 \times 10^{11}}{3} \times \left(\frac{\pi \times 0.1^4}{32}\right) = 2.716 \times 10^5\ Nm/rad$$

We have the equation of natural frequency

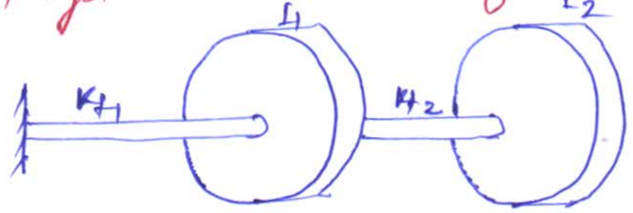
$$\omega_n = \sqrt{\frac{k_t (J_1 + J_2)}{J_1 J_2}} = \sqrt{\frac{2.716 \times 10^5 (98 + 451.25)}{98 \times 451.25}}$$

$$= \boxed{58.08\ rad/sec.}$$

$$f_n = \frac{58.08}{2\pi} = \boxed{9.24\ Hz.}$$

Example

Determine the natural frequencies and mode shapes of the torsional system shown in the figure. Take $I_1 = I$, $I_2 = 2I$ and $k_{t1} = k_{t2} = K$



Let θ_1 and θ_2 be the angular displacements of I_1 and I_2 respectively.

The equations of motion can be written as

$$\left.\begin{aligned} I_1 \ddot{\theta}_1 + k_{t1}\theta_1 + k_{t2}(\theta_1 - \theta_2) &= 0 \\ I_2 \ddot{\theta}_2 + k_{t2}(\theta_2 - \theta_1) &= 0 \end{aligned}\right\} \quad —(1)$$

Rearranging and substituting $I_1 = I$ and $I_2 = 2I$ and $k_{t1} = k_{t2} = K$

$$I_1\ddot{\theta}_1 + \cancel{k_{t1}\theta_1}(k_{t1} + k_{t2})\theta_1 = k_{t2}\theta_2$$

$$I_2\ddot{\theta}_2 + k_{t2}\theta_2 = k_{t2}\theta_1$$

or

$$\left.\begin{aligned} I\ddot{\theta}_1 + 2K\theta_1 &= K\theta_2 \\ 2I\ddot{\theta}_2 + K\theta_2 &= K\theta_1 \end{aligned}\right\} \quad —(2)$$

~~from equation (2) the two ratios obtained are:~~

$$\frac{\theta_1}{\theta_2} = \frac{K}{\quad}$$

Assuming the steady state solution for principal mode of vibration

$$\theta_1 = \beta_1 \sin\omega t \qquad \theta_2 = \beta_2 \sin\omega t \quad -(3)$$

from equation (3)

$$\dot\theta_1 = \omega\beta_1 \cos\omega t \qquad \dot\theta_2 = \omega\beta_2 \cos\omega t$$

$$\ddot\theta_1 = -\omega^2\beta_1 \sin\omega t \qquad \ddot\theta_2 = -\omega^2\beta_2 \sin\omega t \quad -(4)$$

substituting the value of θ in eq. (2)

$$\left.\begin{aligned} -I\omega^2\beta_1 \sin\omega t + 2K\beta_1 \sin\omega t &= K\beta_2 \sin\omega t \\ -2I\omega^2\beta_2 \sin\omega t + K\beta_2 \sin\omega t &= K\beta_1 \sin\omega t \end{aligned}\right\} -(5)$$

or

$$\left.\begin{aligned} (-I\omega^2 + 2K)\beta_1 &= K\beta_2 \\ (-2I\omega^2 + K)\beta_2 &= K\beta_1 \end{aligned}\right\} -(6)$$

from equation (6)

$$\frac{\beta_1}{\beta_2} = \frac{K}{2K - I\omega^2} \quad -(7)$$

$$\frac{\beta_1}{\beta_2} = \frac{K - 2I\omega^2}{K} \quad -(8)$$

Equating eq. (7) and (8)

$$\frac{K}{2K - I\omega^2} = \frac{K - 2I\omega^2}{K}$$

$$\Rightarrow 2K^2 - 4KI\omega^2 - KI\omega^2 + 2I^2\omega^4 = K^2$$

$$\Rightarrow 2I^2\omega^4 - 5KI\omega^2 + K^2 = 0 \quad -(9)$$

solving the equation (9) we have

$$\omega^2 = \frac{5KI \mp \sqrt{(5KI)^2 - 4(2I^2K^2)}}{4I^2}$$

or $$\omega^2 = \frac{5IK \mp \sqrt{17}\cdot KI}{4I^2} \quad -(10)$$

so $\omega_{n_1} = \cancel{(5-\sqrt{17})}\sqrt{(5-\sqrt{17})\frac{K}{4I}}$ rad/s

$\omega_{n_2} = \sqrt{(5+\sqrt{17})\frac{K}{4I}}$ rad/sec

The mode shapes obtained are:

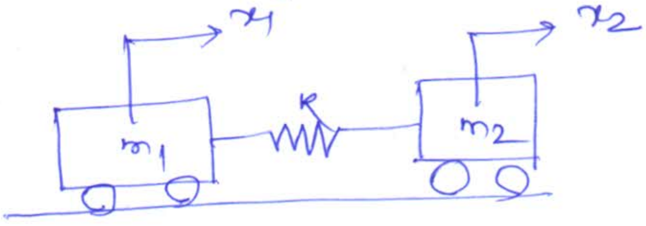
$$\left(\frac{\beta_1}{\beta_2}\right)_1 = \frac{K-2I\omega^2}{K} = 1-(5-\sqrt{17})\frac{1}{2} = 0.561$$

$$\left(\frac{\beta_1}{\beta_2}\right)_2 = \frac{K-2I\omega^2}{K} = 1-(5+\sqrt{17})\frac{1}{2} = -3.561$$

(Ans)

Semi-Definite system:-

When one of the natural frequencies of a system is zero, there is no relative motion in the system and the system moves as a rigid body. Such a systems are called semi-definite systems or un-restrained systems or degenerate systems.



Taking an example as shown in the figure, where two masses m_1 and m_2 are connected with a coupling spring K

The equation of motion of the system can be written as;

$$\left.\begin{aligned} m_1\ddot{x}_1 + K(x_1-x_2) &= 0 \\ m_2\ddot{x}_2 + K(x_2-x_1) &= 0 \end{aligned}\right\} \quad -(1)$$

or

$$\left.\begin{aligned} m_1\ddot{x}_1 + Kx_1 &= Kx_2 \\ m_2\ddot{x}_2 + Kx_2 &= Kx_1 \end{aligned}\right\} \quad -(2)$$

Assuming the motion to be harmonic

$$\left.\begin{aligned} x_1 &= X_1 \sin\omega t \\ x_2 &= X_2 \sin\omega t \end{aligned}\right\} \quad -(3)$$

from equation (3)

$$\dot{x}_1 = \omega X_1 \cos\omega t \qquad \dot{x}_2 = \omega X_2 \cos\omega t$$

$$\ddot{x}_1 = -\omega^2 X_1 \sin\omega t \qquad \ddot{x}_2 = -\omega^2 X_2 \sin\omega t \quad -(4)$$

substituting the values of x in eq. (2)

$$\left.\begin{aligned} -m_1\omega^2 X_1 \sin\omega t + K X_1 \sin\omega t &= K X_2 \sin\omega t \\ -m_2\omega^2 X_2 \sin\omega t + K X_2 \sin\omega t &= K X_1 \sin\omega t \end{aligned}\right\} \quad -(5)$$

Rearranging and removing $\sin\omega t$ from eq. (5)

$$\left.\begin{aligned} (-m_1\omega^2 + K)\, X_1 &= K X_2 \quad -(6) \\ (-m_2\omega^2 + K)\, X_2 &= K X_1 \end{aligned}\right\}$$

The ratio of amplitudes obtained from eq. (6) are

$$\frac{X_1}{X_2} = \frac{K}{(K - m_1\omega^2)} \quad -(7)$$

$$\frac{X_1}{X_2} = \frac{K - m_2\omega^2}{K} \quad -(8)$$

Equating eq. (7) and (8) we have

$$\frac{K}{K - m_1\omega^2} = \frac{K - m_2\omega^2}{K}$$

$$\Rightarrow \cancel{K^2} - K m_2\omega^2 - K m_1\omega^2 + m_1 m_2\omega^4 = \cancel{K^2}$$

$$\Rightarrow m_1 m_2 \omega^4 - (m_1 + m_2) K\omega^2 = 0$$

$$\Rightarrow [m_1 m_2 \omega^2 - (m_1 + m_2) K]\, \omega^2 = 0$$

$$\Rightarrow m_1 m_2 \omega^2 - (m_1 + m_2) K = 0 \quad -(9)$$

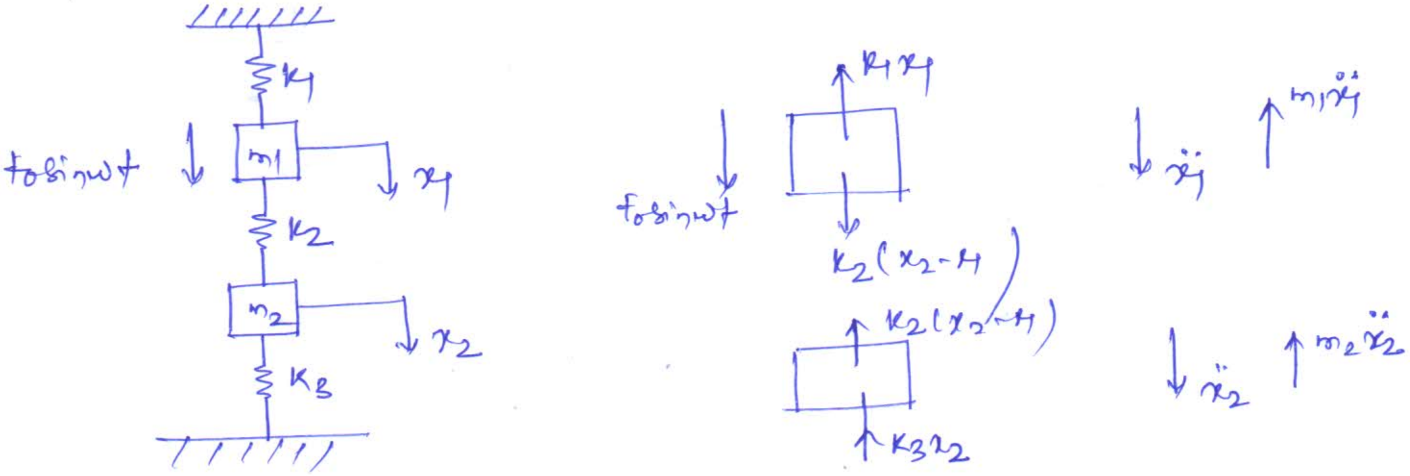
The two values of natural frequencies obtained are:

$$\omega_{n_1} = 0 \qquad \omega_{n_2} = \sqrt{\frac{K(m_1 + m_2)}{m_1 m_2}} \quad -(10)$$

from the analysis it can be seen that one of the natural frequencies is zero and thus the system is ~~not~~ is in semi-definite state.

Undamped forced vibrations with Harmonic Excitation

When a harmonic forcing function act on a system, the solution consists of the transient part and steady state part.



The equation of motion of the system can be expressed as:

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) &= f_0 \sin\omega t \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3x_2 &= 0 \end{aligned}\right\} \quad -(1)$$

or,

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 &= f_0 \sin\omega t \\ m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 &= 0 \end{aligned}\right\} \quad -(2)$$

Assuming a steady solution

$$\left.\begin{aligned} x_1 &= X_1 \sin\omega t \\ x_2 &= X_2 \sin\omega t \end{aligned}\right\} \quad -(3)$$

from equation (3)

$$\dot{x}_1 = \omega X_1 \cos\omega t \qquad \dot{x}_2 = \omega X_2 \cos\omega t$$

$$\ddot{x}_1 = -\omega^2 X_1 \sin\omega t \qquad \ddot{x}_2 = -\omega^2 X_2 \sin\omega t \quad -(4)$$

substituting the values in equation (2)

$$\left.\begin{aligned} -m_1\omega^2X_1\sin\omega t + (k_1+k_2)X_1\sin\omega t - k_2X_2\sin\omega t &= f_0\sin\omega t \\ -m_2\omega^2X_2\sin\omega t + (k_2+k_3)X_2\sin\omega t - k_2X_1\sin\omega t &= 0 \end{aligned}\right\} \quad -(5)$$

or

$$\left.\begin{aligned} \left[-m_1\omega^2 + (k_1+k_2)\right]X_1 - k_2X_2 &= f_0 \\ k_2X_1 - \left[-m_2\omega^2 + (k_2+k_3)\right]X_2 &= 0 \end{aligned}\right\} \quad -(6)$$

from eq. (6) we have

$$x_2 = \frac{k_2 x_1}{\left[-m_2 w^2 + (k_2+k_3)\right]} \quad -(7)$$

substituting the value of x_2 in eq. (6) we have.

$$\left[-m_1 w^2 + (k_1+k_2)\right] x_1 - k_2\left[\frac{k_2 x_1}{-m_2 w^2 + (k_2+k_3)}\right] = F_0$$

$$\Rightarrow \cancel{-m_1 w^2 + (k_1+k_2)x_1 - \frac{k_2^2 x_1}{-m_2 w^2 + (k_2+k_3)} = F_0}$$

$$\Rightarrow \cancel{-m_1 w^2 + \left[(k_1+k_2) - \frac{k_2^2}{-m_2 w^2 + (k_2+k_3)}\right] x_1 = F_0}$$

$$\left[-m_1 w^2 + (k_1+k_2)\right] x_1 - \left[\frac{k_2^2 x_1}{-m_2 w^2 + (k_2+k_3)}\right] = F_0$$

$$\Rightarrow \left[\left\{-m_1 w^2 + (k_1+k_2)\right\}\left\{-m_2 w^2 + (k_2+k_3)\right\} - k_2^2\right] x_1 = F_0\left[-m_2 w^2 + (k_2+k_3)\right]$$

$$\text{or} \quad x_1 = \frac{F_0\left[(k_2+k_3) - m_2 w^2\right]}{\left[m_1 m_2 w^4 - (k_2+k_3) m_1 w^2 - (k_1+k_2) m_2 w^2 + k_1 k_2 + k_1 k_3 + k_2^2 + k_2 k_3 - k_2^2\right]}$$

$$\text{or} \quad \boxed{x_1 = \frac{F_0\left[(k_2+k_3) - m_2 w^2\right]}{m_1 m_2 w^4 - \left\{m_1(k_2+k_3) + m_2(k_1+k_2)\right\} w^2 + (k_1 k_2 + k_1 k_3 + k_2 k_3)}} \quad -(8)$$

$$x_2 = \frac{k_2 \cdot F_0 \cancel{\left[(k_2+k_3) - m_2 w^2\right]}}{\cancel{\left[(k_2+k_3) - m_2 w^2\right]}\left[m_1 m_2 w^4 - \left\{m_1(k_2+k_3) + m_2(k_1+k_2)\right\} w^2 + \left\{k_1 k_2 + k_1 k_3 + k_2 k_3\right\}\right]}$$

$$\text{or} \quad \boxed{x_2 = \frac{k_2 F_0}{m_1 m_2 w^4 - \left\{m_1(k_2+k_3) + m_2(k_1+k_2)\right\} w^2 + \left\{k_1 k_2 + k_1 k_3 + k_2 k_3\right\}}} \quad -(9)$$

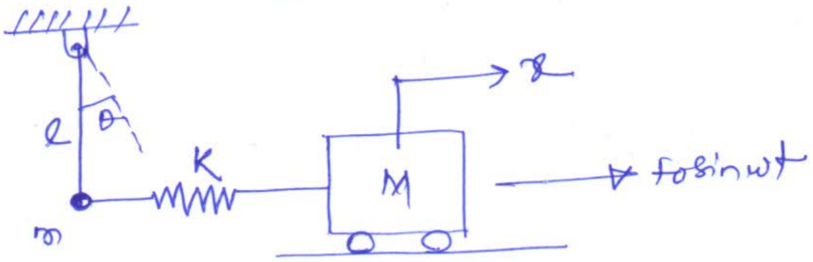
It is observed that the denominators of eq. (8) and (9) are identical. comparing the denominators of eq. (8) and (9) with the frequency equation, it is seen that whenever the excitation frequency w becomes equal to any of the two natural frequencies w_{n_1} and w_{n_2}, the amplitudes

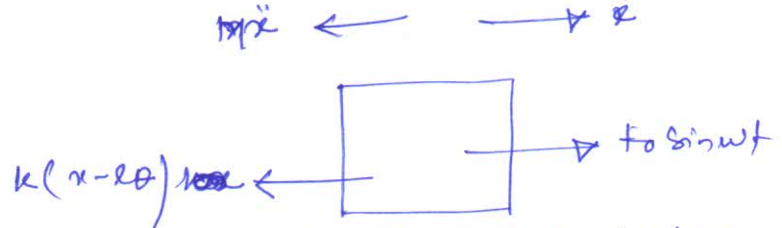
X_1 and X_2 become infinite, which is a resonance condition. Thus we have two resonance frequencies each corresponding to the natural frequencies of the system. Also X_1 becomes zero when $\omega = \sqrt{(k_2+k_3)/m_2}$, thereby making mass m_1 motionless at this frequency. Such conditions are not applicable for mass m_2.

- The mass which is excited can have zero amplitude of vibration under certain conditions by coupling it to another spring-mass system forms the principle of dynamic vibration absorber.

Example

For a system shown in the figure find the steady state amplitude of the mass M under the exciting force $F_0 \sin\omega t$. Is there any frequency at which the amplitude of the mass is (i) zero, (ii) infinity.





The equation of motion of the system

$$\left.\begin{aligned} M\ddot{x} + k(x - l\theta) &= F_0 \sin\omega t \\ ml\ddot{\theta} - k(x - l\theta) + mg\theta &= 0 \end{aligned}\right\} \quad -(1)$$

Let the steady solution be

$$\left.\begin{aligned} x &= X \sin\omega t \\ \theta &= \beta \sin\omega t \end{aligned}\right\} \quad -(2)$$

from eq. (2) $\dot{x} = \omega X \cos\omega t$ | $\dot{\theta} = \omega\beta \cos\omega t$

$\ddot{x} = -\omega^2 X \sin\omega t$ | $\ddot{\theta} = -\omega^2 \beta \sin\omega t$ $\quad -(3)$

substituting the value of $x, \ddot{x}, \theta, \ddot{\theta}$ in eq. (1)

$$\left.\begin{aligned} &-M\omega^2 X\sin\omega t + k(X\sin\omega t - l\beta\sin\omega t) = f_0\sin\omega t \\ &-m\omega^2\beta l\sin\omega t - k(X\sin\omega t - l\beta\sin\omega t) + mg\beta\sin\omega t = 0 \end{aligned}\right\} \quad -(4)$$

or $\quad (-M\omega^2 + k)X - kl\beta = f_0$

$\cancel{(-m\omega^2 + k)\beta l - kX = 0}$

$(-m\omega^2 - k)\beta l + mg\beta - kX = 0$

or $\quad (M\omega^2 - k)X + kl\beta = -f_0 \quad -(5)$

$\quad (ml\omega^2 + kl + mg)\beta + kX = 0 \quad -(6)$

from eq. (6)

$$\beta = \frac{-kX}{(ml\omega^2 + kl + mg)} \quad -(7)$$

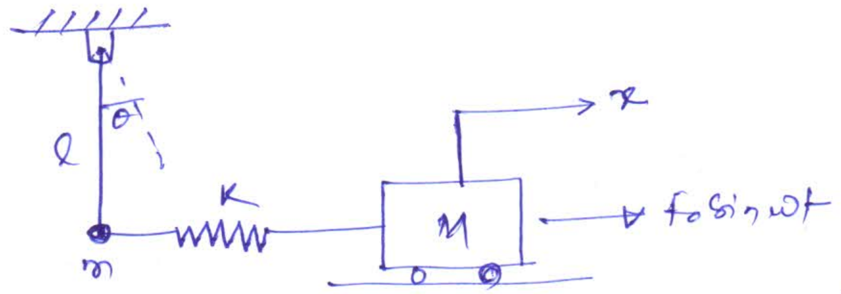
substituting the value of β in eq (5)

$$(M\omega^2 - k)X - \frac{k^2 l X}{(ml\omega^2 + kl + mg)} = -f_0$$

or $\quad \{(M\omega^2 - K)(ml\omega^2 + Kl + mg) - K^2 l\}X = -f_0(ml\omega^2 + kl + mg$

(9.4)

Example For the system shown in the fig, find the steady state amplitude of the mass M under the exciting force $F_0 \sin \omega t$. Is there any frequency at which the amplitude of the mass is (i) zero, (ii) infinity?



Considering small amplitudes of vibration, the equations of motion for the system may be written as:

$$\left.\begin{aligned} M\ddot{x} &= -K(x - l\theta) + F_0 \sin \omega t \\ ml\ddot{\theta} &= K(x - l\theta) - mg\theta \end{aligned}\right\} \quad -(1)$$

Assuming a steady state solution of

$$\begin{aligned} x &= X \sin \omega t \\ \theta &= \beta \sin \omega t \end{aligned} \quad -(2)$$

We have $\dot{x} = \omega X \cos \omega t$ | $\dot{\theta} = \omega \beta \cos \omega t$

$\ddot{x} = -\omega^2 X \sin \omega t$ | $\ddot{\theta} = -\omega^2 \beta \sin \omega t \quad -(3)$

Substituting the values of $x, \ddot{x}, \theta$ and $\ddot{\theta}$ in equation (1)

$$\left.\begin{aligned} -M\omega^2 X \sin \omega t + K(X \sin \omega t - l\beta \sin \omega t) &= F_0 \sin \omega t \\ -m\omega^2 \beta l \sin \omega t - K(X \sin \omega t - l\beta \sin \omega t) + mg\beta \sin \omega t &= 0 \end{aligned}\right\} \quad -(4)$$

or

$$\left.\begin{aligned} -M\omega^2 X \sin \omega t + KX \sin \omega t - Kl\beta \sin \omega t &= F_0 \sin \omega t \\ -m\omega^2 \beta l \sin \omega t - KX \sin \omega t + Kl\beta \sin \omega t + mg\beta \sin \omega t &= 0 \end{aligned}\right\} \quad -(5)$$

or

$$\begin{aligned} (-M\omega^2 + K) X - Kl\beta &= F_0 \\ -KX - (m\omega^2 l - Kl - mg)\beta &= 0 \end{aligned}$$

or

$$\left.\begin{aligned} (M\omega^2 - K) X + Kl\beta &= -F_0 \\ KX + (m\omega^2 l - Kl - mg)\beta &= 0 \end{aligned}\right\} \quad -(6)$$

Solving the equations we have

$$X = \frac{F_0\left[Kl + mg - m_1 l\omega^2\right]}{Mml\omega^4 - \left[(M+m)Kl + Mmg\right]\omega^2 + kmg} \quad \text{---(4)}$$

(i) amplitude is zero at

$$\omega = \sqrt{\frac{Kl+mg}{ml}}$$

Natural frequency of the system when mass M is considered to be fixed.

(ii) amplitude is infinity when the denominator is equal to zero.

Vibration Absorbers :-

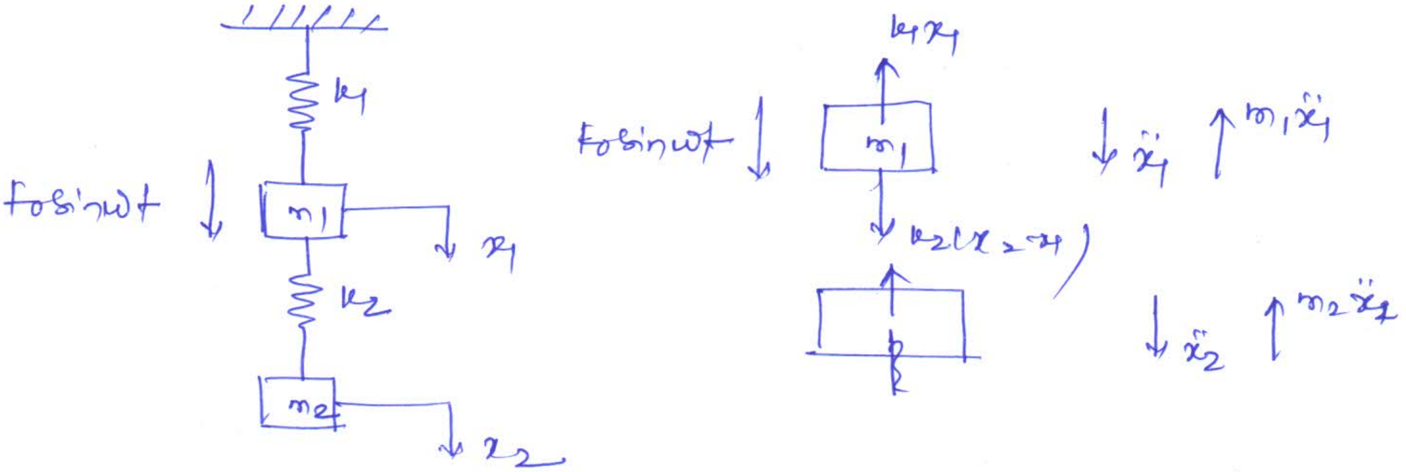
When a machine or a system is subjected to an external excitation force whose excitation frequency nearly coincide with the natural frequency of the machine or system, excessive vibrations are induced in the system. Such vibrations may be eliminated by coupling a properly designed auxiliary spring-mass system to the main system. This auxiliary spring-mass system is called dynamic vibration absorber.

- This type of absorber is extreamly effective at one speed only thus is suitable only for constant speed machines. A damped dynamic vibration absorber can take care of the entire frequency range of excitation but at the cost of reduced effectiveness.

Undamped dynamic Vibration absorber :-

The undamped dynamic vibration absorber is also called Frahm vibration absorber.

- The principle of undamped dynamic vibration absorber can be analysed by taking a two dof spring mass system.



The differential equation of motion may be written as:

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2-x_1) &= F_0 \sin\omega t \\ m_2\ddot{x}_2 + k_2(x_2-x_1) &= 0 \end{aligned}\right\} \quad \text{--- (1)}$$

Assuming a steady state solution

$$x_1 = X_1 \sin\omega t \qquad x_2 = X_2 \sin\omega t \quad \text{--- (2)}$$

so,

$$\begin{aligned} \dot{x}_1 &= \omega X_1 \cos\omega t & \dot{x}_2 &= \omega X_2 \cos\omega t \\ \ddot{x}_1 &= -\omega^2 X_1 \sin\omega t & \ddot{x}_2 &= -\omega^2 X_2 \sin\omega t \end{aligned} \quad \text{--- (3)}$$

substituting the values in equation (1)

$$-m_1\omega^2 X_1 \sin\omega t + k_1X_1 \sin\omega t - k_2(X_2-X_1)\sin\omega t = F_0 \sin\omega t$$

$$-m_2\omega^2 X_2 \sin\omega t + k_2(X_2-X_1)\sin\omega t = 0$$

or $\left[-m_1\omega^2 + (k_1+k_2)\right]X_1 - k_2X_2 = F_0$ — (4)

$\left[-m_2\omega^2 + k_2\right]X_2 - k_2X_1 = 0$ — (5)

from equation (5)

$$\cancel{x_1 = \frac{(k_2 - m_2\omega^2)\,x_2}{k_2}}$$

~~substituting the value of x_1 in eq (4)~~

$$\cancel{\left[-m_1\omega^2 + (k_1+k_2)\right]\left[\frac{(k_2 - m_2\omega^2)\,x_2}{k_2}\right] - k_2 x_2 = F_0}$$

$$\Rightarrow \quad x_2 = \frac{k_2 x_1}{(k_2 - m_2\omega^2)} \qquad \text{--- (6)}$$

Substituting the value of x_2 in eq. (4)

$$\left[(k_1+k_2) - m_1\omega^2\right]x_1 - \frac{k_2^2\,x_1}{(k_2 - m_2\omega^2)} = F_0$$

$$\Rightarrow \left[\left\{(k_1+k_2) - m_1\omega^2\right\}\left\{k_2 - m_2\omega^2\right\} - k_2^2\right]x_1 = F_0(k_2 - m_2\omega^2)$$

$$\Rightarrow \left(k_1k_2 - m_2k_1\omega^2 + \cancel{k_2^2} - m_2k_2\omega^2 - m_1k_2\omega^2 + m_1m_2\omega^4 - \cancel{k_2^2}\right)x_1 = F_0(k_2 - m_2\omega^2)$$

$$\Rightarrow \left[m_1m_2\omega^4 - \left\{m_1k_2 + m_2(k_1+k_2)\right\}\omega^2 + k_1k_2\right]x_1 = F_0(k_2 - m_2\omega^2)$$

$$\Rightarrow x_1 = \frac{F_0(k_2 - m_2\omega^2)}{\left[m_1m_2\omega^4 - \left\{m_1k_2 + m_2(k_1+k_2)\right\}\omega^2 + k_1k_2\right]} \qquad \text{--- (7)}$$

$$x_2 = \frac{k_2F_0}{\left[m_1m_2\omega^4 - \left\{m_1k_2 + m_2(k_1+k_2)\right\}\omega^2 + k_1k_2\right]} \qquad \text{--- (8)}$$

To bring these equations to a dimensionless forms, dividing the numerators and denominators by k_1k_2 and introducing the following notations.

$x_{st} = F_0/k_1$ = zero frequency deflection

$\omega_1 = \sqrt{\frac{k_1}{m_1}}$ = natural frequency of main system

$\omega_2 = \sqrt{\frac{k_2}{m_2}}$ = natural frequency of absorber alone

$\mu = m_2/m_1$ = ratio of absorber mass to the main mass.

(98)

Equation (7) and (8) can be written as a dimensionless form.

$$\left(\frac{X_1}{X_{st}}\right) = \frac{\left(1-\frac{\omega^2}{\omega_2^2}\right)}{\frac{\omega^4}{\omega_1^2\omega_2^2} - \left[(1+\mu)\frac{\omega^2}{\omega_1^2} + \frac{\omega^2}{\omega_2^2}\right] + 1} \quad \text{---(9)}$$

$$\left(\frac{X_2}{X_{st}}\right) = \frac{1}{\frac{\omega^4}{\omega_1^2\omega_2^2} - \left[(1+\mu)\frac{\omega^2}{\omega_1^2} + \frac{\omega^2}{\omega_2^2}\right] + 1} \quad \text{---(10)}$$

Eq. (9) indicates $X_1 = 0$ when $\omega = \omega_2$ i.e., when the excitation frequency is equal to the natural frequency of absorber, the amplitude of main system becomes zero even though it is excited by a harmonic force.

- Now substituting $\omega = \omega_2$ in eq (10)

$$\cancel{X_2 = \frac{X_{st}}{\mu\frac{\omega_2^2}{\omega_1^2}}}$$

$$\frac{X_2}{X_{st}} = \frac{1}{\frac{\omega_2^2}{\omega_1^2} - \left[(1+\mu)\frac{\omega_2^2}{\omega_1^2} + 1\right] + 1}$$

$$= \frac{1}{\frac{\omega_2^2}{\omega_1^2} - \frac{\omega_2^2}{\omega_1^2} - \mu\frac{\omega_2^2}{\omega_1^2} - 1 + 1}$$

$$\Rightarrow X_2 = \frac{-X_{st}}{\mu\frac{\omega_2^2}{\omega_1}} = -\frac{F_0/K_1}{\frac{m_2}{m_1}\cdot\frac{k_2}{\cancel{m_2}}\cdot\frac{m_1}{k_1}}$$

$$\left.\begin{aligned}\Rightarrow X_2 &= -F_0/k_2 \\ \text{or } F_0 &= -k_2/X_2\end{aligned}\right\} \quad \text{---(11)}$$

Eq. (11) indicates the spring force $k_2 X_2$ on the main mass due to amplitude X_2 of the absorber is equal and opposite to the exciting force on the main mass, so the main system vibrations have been reduced to zero and these vibrations have been taken by the absorber.

- Addition of a vibration absorber to main system is not much meaningful unless the main system is operating at resonance or at least near to it. Under these conditions we have $\omega = \omega_1$
- But for the absorber to be effective we ~~need to h~~ already have $\omega_2 = \omega$.

Therefore for the effectiveness of the absorber at operating frequency corresponding to the natural frequency of the main system alone, we have

$$\omega_2 = \omega_1 \quad \text{or} \quad \boxed{\frac{k_2}{m_2} = \frac{k_1}{m_1}}$$

When the above condition is fulfilled, the absorber is known to ~~have~~ be tuned absorber.

MODULE 3

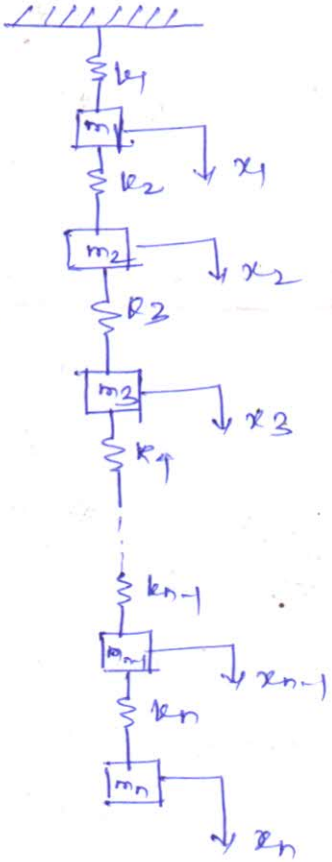
MODULE - III

Multi Degree of freedom system :-

Multi degree of freedom systems are defined as those systems which require two or more coordinates to describe their motion.

Equation of Motion :-

An undamped system having n dof is shown in the figure.



Let $x_1, x_2, x_3, \dots x_n$ = displacement of respective masses at any instance.

Then the differential equation of motion for each mass can be expressed using Newton's second law of motion as.

$$\left.\begin{aligned}
&m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0\\
&m_2\ddot{x}_2 - k_2(x_1 - x_2) + k_3(x_2 - x_3) = 0\\
&m_3\ddot{x}_3 - k_3(x_2 - x_3) + k_4(x_3 - x_4) = 0\\
&- - - - - - - - -\\
&m_n\ddot{x}_n - k_n(x_{n-1} - x_n) = 0
\end{aligned}\right\} \quad \text{--- (1)}$$

or

$$\left.\begin{aligned}
&m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0\\
&m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0\\
&m_3\ddot{x}_3 - k_3x_2 + (k_3 + k_4)x_3 - k_4x_4 = 0\\
&- - - - - - - - -
\end{aligned}\right\} \quad \text{--- (2)}$$

$$m_n\ddot{x}_n - k_nx_{n-1} + k_nx_n = 0$$

Equation (2) can be expressed in a matrix form as.

$$\begin{bmatrix} m_1 & 0 & 0 & 0\\ 0 & m_2 & 0 & 0\\ 0 & 0 & m_3 & 0\\ - & - & - & -\\ 0 & 0 & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\\ \vdots\\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & \cdots & 0\\ -k_2 & (k_2+k_3) & -k_3 & \cdots & 0\\ 0 & -k_3 & (k_3+k_4) & \cdots & 0\\ - & - & - & & \\ 0 & 0 & 0 & \cdots & k_n \end{bmatrix} \begin{Bmatrix} x_1\\ x_2\\ x_3\\ \vdots\\ x_n \end{Bmatrix} = \begin{Bmatrix} 0\\ 0\\ 0\\ \vdots\\ 0 \end{Bmatrix}$$

$$\text{--- (3)}$$

or $$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \quad \text{— (4)}$$

Where

$[M] \rightarrow$ Mass matrix of n^{th} order.

$[K] \rightarrow$ stiffness matrix of n^{th} order.

$\{x\} \rightarrow$ column matrix of n elements, corresponding to the dynamic displacement of respective n masses.

And eq. (4) is similar to

$m\ddot{x} + kx = 0$, (i.e. eq. of a single dof system)

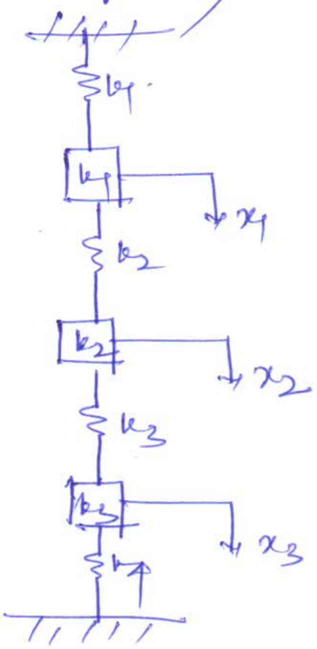
Natural Frequencies and Mode shapes (for a 3 dof system)

~~Assuming a steady state solution of~~

$\cancel{x_1 = X_1 \sin\omega t}$

$\cancel{x_2 = X_2 \sin\omega t}$

$\cancel{x_3 = X_3 \sin\omega t}$



The equation of motion

$$\left.\begin{aligned}
m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) &= 0\\
m_2\ddot{x}_2 - k_2(x_1 - x_2) + k_3(x_2 - x_3) &= 0\\
m_3\ddot{x}_3 - k_3(x_2 - x_3) + k_4x_3 &= 0
\end{aligned}\right\} \quad \text{— (1)}$$

or

$$\left.\begin{aligned}
m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= 0\\
m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 &= 0\\
m_3\ddot{x}_3 - k_3x_2 + (k_3 + k_4)x_3 &= 0
\end{aligned}\right\} \quad \text{— (2)}$$

In matrix form

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & (k_3+k_4) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0 \quad \text{— (3)}$$

Assuming a steady solution of

$$\left.\begin{aligned}
x_1 &= X_1 \sin\omega t\\
x_2 &= X_2 \sin\omega t\\
x_3 &= X_3 \sin\omega t
\end{aligned}\right\} \quad \text{— (4)}$$

Substituting the values of $x_1, \ddot{x}_1, x_2, \ddot{x}_2, x_3, \ddot{x}_3$ in eq.(2)

$$\left.\begin{aligned} &[-m_1\omega^2 + (k_1+k_2)]X_1 - k_2X_2 = 0 \\ &-k_2X_1 + [-m_2\omega^2 + (k_2+k_3)]X_2 - k_3X_3 = 0 \\ &-k_3X_2 + [-m_3\omega^2 + (k_3+k_4)]X_3 = 0 \end{aligned}\right\} \quad -(5)$$

The characteristic frequency equation can be obtained by the determinant as under

$$\begin{vmatrix} (-m_1\omega^2 + k_1+k_2) & -k_2 & 0 \\ -k_2 & (-m_2\omega^2+k_2+k_3) & -k_3 \\ 0 & -k_3 & (-m_3\omega^2+k_3+k_4) \end{vmatrix} = 0 \quad -(6)$$

Expanding the above determinant and solving for ω^2, three natural frequencies of the 3 dof system can be obtained. Also the amplitude ratios for obtaining the principal modes of vibration can be obtained as {from eq. (5)}

$$\cancel{\left(\frac{X_2}{X_1}\right) = \frac{k_1+k_2-m_1\omega^2}{k_2}}$$

Matrix Method :-

Matrix method is a widely used method to determine the natural frequency of a multi dof system. The advantage of this method is that a computer programme can be developed to solve the equation and generate the eigen values and eigen vectors directly with much ease.

We know that for a multi dof system the equation of motion in matrix form can be expressed as:

$$[M]\{\ddot{x}\} + [k]\{x\} = 0 \quad -(1)$$

multiplying eq. (1) by $[m]^{-1}$, we have

$$[I]\{\ddot{x}\} + [C]\{x\} = 0 \quad -(2)$$

Where $[m]^{-1}[m] = [I]$, a unit matrix

$[m]^{-1}[k] = [C]$, a dynamic matrix

The value of $[m]^{-1}$ can be obtained as $[m]^{-1} = \frac{Adj[m]}{|m|}$

Assuming harmonic oscillation of frequency ω,

$$\{x\} = \{X\} \sin\omega t$$

We have $\{\ddot{x}\} = -\omega^2 \{x\} = -\lambda \{X\} \sin\omega t$

Where $\lambda = \omega^2$

The equation (2) can be written as

$$-\lambda [I]\{x\} + [C]\{x\} = 0$$

or $$\big[[C] - \lambda [I]\big]\{x\} = 0$$

or $$\big[\lambda [I] - [C]\big]\{x\} = 0 \quad -(3)$$

The characteristics equation or frequency equation is given by

$$\boxed{\big|\lambda [I] - [C]\big| = 0} \quad -(4)$$

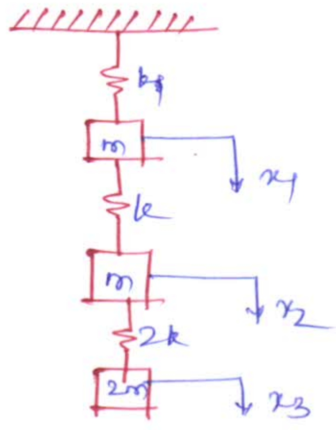
The roots of the frequency equation (λ_i) are called eigen values and square root of these quantities are the system natural frequencies i.e

$$\boxed{\omega_i = \sqrt{\lambda_i}} \quad -(5)$$

Once the eigen values are obtained these can be substituted in eq. (3) to find mode shapes, called eigen vectors.

Example!-

For the 3 dof system shown in the figure find the natural frequencies,



The equations of motion may be written as:

$$\left.\begin{aligned} m\ddot{x}_1 + kx_1 + k(x_1 - x_2) &= 0 \\ m\ddot{x}_2 - k(x_1 - x_2) + 2k(x_2 - x_3) &= 0 \\ 2m\ddot{x}_3 - 2k(x_2 - x_3) &= 0 \end{aligned}\right\} \quad -(1)$$

or

$$\left.\begin{aligned} m\ddot{x}_1 + 2kx_1 - kx_2 &= 0 \\ m\ddot{x}_2 - kx_1 + 3kx_2 - 2kx_3 &= 0 \\ 2m\ddot{x}_3 - 2kx_2 + 2kx_3 &= 0 \end{aligned}\right\} \quad -(2)$$

In matrix form

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0 \quad -(3)$$

or $[m]\{\ddot{x}\} + [k]\{x\} = 0$

Dynamic matrix $[C] = [m]^{-1}[k]$

and $[m]^{-1} = \dfrac{\text{Adj}[m]}{|m|}$

$|m| = 2m^3$

$$\text{Adj}[m] = \begin{bmatrix} 2m^2 & 0 & 0 \\ 0 & 2m^2 & 0 \\ 0 & 0 & m^2 \end{bmatrix}$$

Therefore $[m]^{-1} = \dfrac{\text{Adj}[m]}{|m|} = \begin{bmatrix} 1/m & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/2m \end{bmatrix}$

The dynamic matrix is given by

$$C = [m]^{-1}[K] = \begin{bmatrix} 1/m & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/2m \end{bmatrix} \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2k}{m} & -k/m & 0 \\ -k/m & 3k/m & -2k/m \\ 0 & -2k/m & k/m \end{bmatrix}$$

Therefore $[\lambda[I] - [C]]$ can be written as

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2k/m & -k/m & 0 \\ -k/m & 3k/m & -2k/m \\ 0 & -k/m & k/m \end{bmatrix} = 0$$

or
$$\begin{vmatrix} \lambda - 2k/m & k/m & 0 \\ k/m & \lambda - 3k/m & 2k/m \\ 0 & k/m & \lambda - k/m \end{vmatrix} = 0$$

or $\lambda^3 - 6\lambda^2 \frac{k}{m} + 8\lambda \frac{k^2}{m^2} - \frac{k^3}{m^3} = 0$

solving the above equation we have

$\lambda_1 = 0.139\ k/m$

$\lambda_2 = 1.746\ k/m$

$\lambda_3 = 4.115\ k/m$,

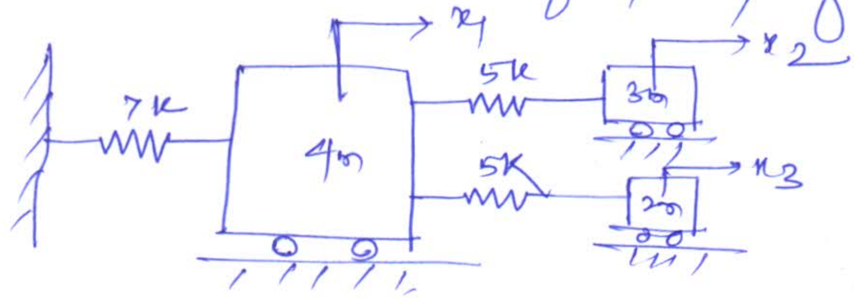
so the natural frequencies are

$\omega_1 = \sqrt{\lambda_1} = 0.373\sqrt{k/m}$

$\omega_2 = \sqrt{\lambda_2} = 1.32\sqrt{k/m}$ (Ans)

$\omega_3 = \sqrt{\lambda_3} = 2.03\sqrt{k/m}$

Example find the natural frequency of the system



Influence coefficients:-

The differential equation of motion of a multi dof system can be expressed in matrix form, which includes mass matrix [M] and stiffness matrix [K]. In case of damping, there will be a damping matrix [C] in the equation

- The equation can also be expressed in terms of flexibility matrix [A] instead of stiffness matrix [K]. ~~flexibility~~

(i) stiffness influence coefficients:-

Flexibility matrix is inverse of stiffness matrix

$$[A] = [K]^{-1}$$

which also means that $[K] = [A]^{-1}$ — (1)

or stiffness $= \dfrac{1}{\text{flexibility}}$

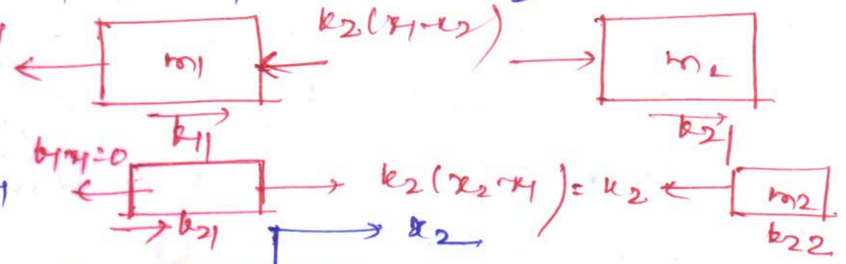
The elements k_{ij}, a_{ij} and c_{ij} of stiffness, flexibility and damping matrices, respectively are referred to as influence coefficients.

The use of influence coefficients facilitate the expression of differential equation of motion of a multi dof system in matrix form,

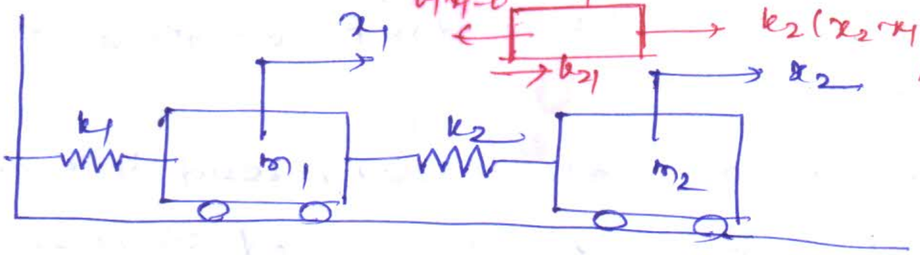
A matrix of stiffness influence coefficient is expressed as:

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ \hline k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix} \quad \text{— (2)}$$

~~Let in k_{ij}~~ The stiffness influence coefficient k_{ij} denotes force at point i due to unit displacement at point j, when all other points are fixed.



Example



Let x_1 and x_2 → displacement of mass m_1 and m_2 respectively

The stiffness influence coefficient k_{ij} can be determined in terms of spring stiffness k_1 and k_2

~~Let $x_1 = 1$ unit~~ while position of mass m_2, $x_2 = 0$.

Writing the differential equation of motion

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1x_1 + k_2(x_1-x_2) &= 0 \\ m_2\ddot{x}_2 - k_2(x_1-x_2) &= 0 \end{aligned}\right\} \quad -(1)$$

or

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 - k_2x_1 + k_2x_2 &= 0 \end{aligned}\right\} \quad -(2)$$

In matrix form

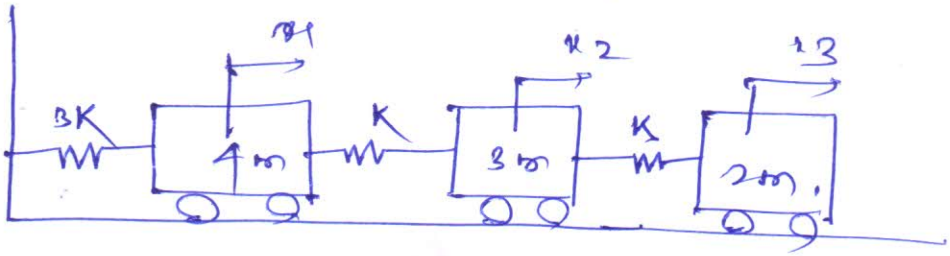
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

So stiffness influence coefficient

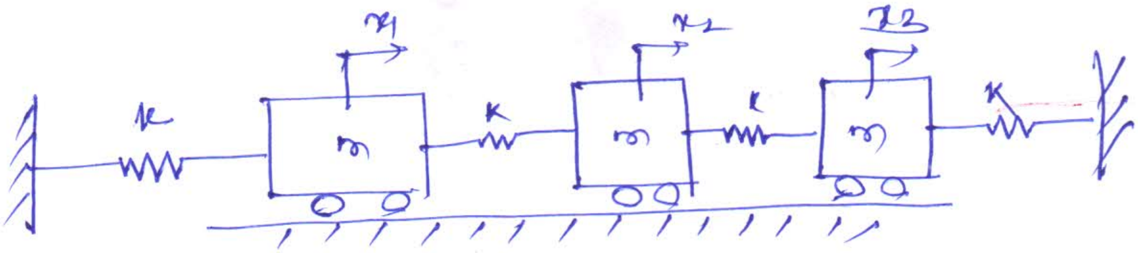
$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & K_2 \end{bmatrix} \quad \text{Ans,}$$

and $[A] = [k]^{-1}$

Problem Determine the value of influence coefficients for the ~~two~~ system shown in the fig



Q.2



Determine the stiffness influence coefficient of the three dof system

ii) Flexibility Influence coefficients :-

If two points i and j of a system are considered, then a_{ij} is defined as the flexibility influence coefficient, which is defined as the deflection at point i due to unit load at point j of the system. The elements of the matrix are called flexibility influence coefficients and a_{ii}, a_{jj} are called the direct influence coefficients and a_{ij}, a_{ji} etc are the cross influence coefficients.

The matrix of flexibility coefficients are :

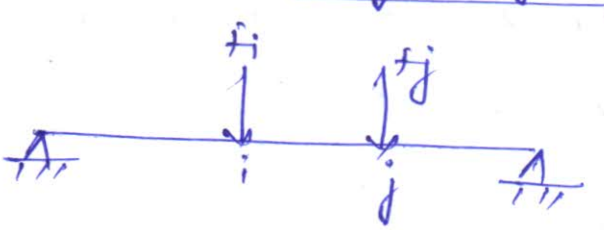
$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ - & - & - & - & - & - \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \quad \text{---(1)}$$

Maxwell Reciprocal theorem :-

It states that the deflection at any point in the system due to a unit load acting at any other point of the system is equal to the deflection at the second point due to unit load acting at the first point

i.e

$$\boxed{a_{ij} = a_{ji}} \quad \text{---(2)}$$



Considering a system and two points i and j in the system

(i) First apply load f_i at point i raising it's value gradually from zero to full value, then apply f_j in same manner, while load f_i acting at point i.

(ii) first apply load f_j at point j gradually from zero to maximum value and then apply f_i at point in same manner with load f_j acting at point j all the time.

As the final deflection curve would be same, so the workdone or strain energy due to these two loads will be same, independent of which load was applied first.

The workdone in case (i) is

$$W_i = \frac{1}{2}(a_{ii} f_i) f_i + \frac{1}{2}(a_{jj} f_j) f_j + (a_{ij} f_i) f_j \quad -(3)$$

Similarly the workdone in second case is

$$W_{ii} = \frac{1}{2}(a_{jj} f_j) f_j + \frac{1}{2}(a_{ii} f_i) f_i + (a_{ji} f_j) f_i \quad -(4)$$

Equating the above two equations we have

$$\boxed{a_{ij} = a_{ji}} \quad -(5)$$

<u>Matrix Iteration Method</u>

This method is used to find the natural frequencies and mode shapes of a multi dof system. for a multi dof system the governing equation can be reduced to the ~~state~~ eigen value problem by

$$\{\ddot{x}\} + [C]\{x\} = 0 \quad -(6)$$

where $[C] = [m]^{-1}[K]$, a dynamic matrix.

so $[C]\{x\} - \omega^2\{x\} = 0$

or $[C]\{x\} = \lambda\{x\} \quad -(7)$

Matrix Iteration Method:-

This method is suitable amongst other iterative methods for determining the lowest eigenvalues (natural frequencies) and eigen vectors (mode shapes) of a multi degree of freedom system.

- The advantage of this method is that the iterative method here results in the principal mode of vibration of the system and corresponding natural frequency, simultaneously, whereas in case of polynomial method separate operation is required to find both.

The equation of motion in ~~case of~~ terms of flexibility matrix can be represented as:

$$[A][M]\{\ddot{x}\}+\{x\}=0 \quad \text{--- (1)}$$

Using a solution $\{x\}=\{X\}\sin\omega t$

We have $\{\dot{x}\}=\omega\{X\}\cos\omega t$

$\{\ddot{x}\}=-\omega^2\{X\}\sin\omega t$

Substituting the value of $\{x\}$ and $\{\ddot{x}\}$ in eq.(1) we have,

$$-\omega^2\{X\}[A][M]+\{X\}=0$$

or $\{X\}=\omega^2[A][M]\{X\}$

or $$\boxed{\{X\}=\omega^2[B]\{X\}} \quad \text{--- (2)}$$

Where $[B]=[A][M]$

Eq.(2) is in the form of

$$\begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix} = \omega^2 \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix} \quad \text{--- (3)}$$

The process is then started by calculating a set of deflection for the right column and then expanding the right hand side

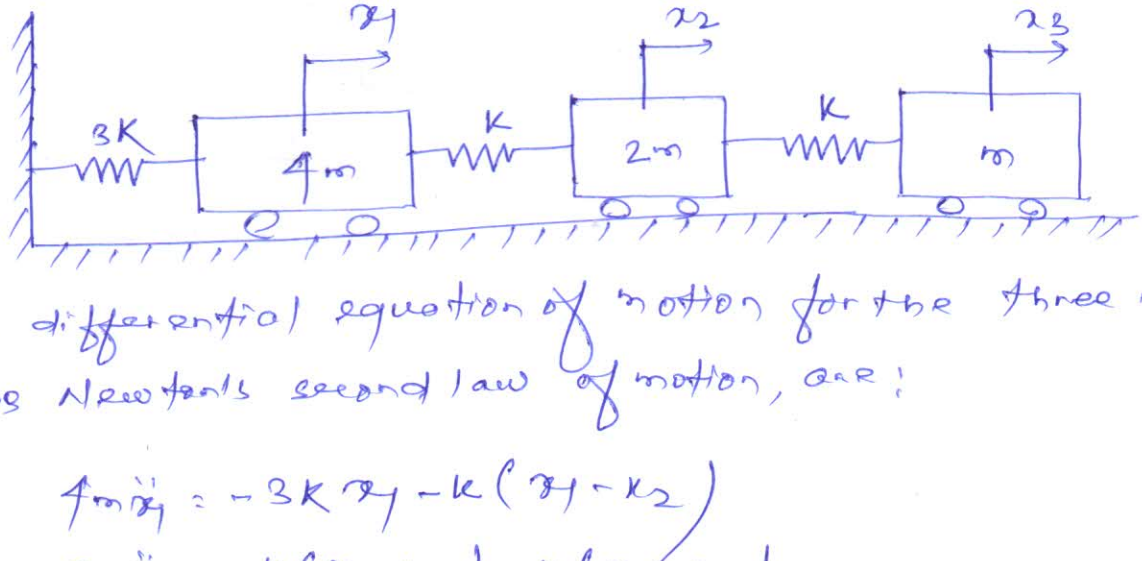
The process is continued until the first mode repeats.

The iteration process with the use of eq. (3) converges to the lowest value of ω^2 so that the fundamental mode of vibration is obtained.

- To obtain the next higher modes and the natural frequencies the orthogonality principle is applied to obtain a modified matrix equation which does not contain the lower modes, and the iterative process is repeated as before.

Example :-

Find the fundamental natural frequency and corresponding mode shape for the system shown in the fig. using matrix iteration method. Also obtain the higher modes using principle of orthogonality.



the differential equation of motion for the three masses using Newton's second law of motion, are:

$$4m\ddot{x}_1 = -3kx_1 - k(x_1 - x_2)$$
$$2m\ddot{x}_2 = k(x_1 - x_2) - k(x_2 - x_3)$$
$$m\ddot{x}_3 = k(x_2 - x_3)$$

Rearranging the equations

$$\left.\begin{aligned} &4m\ddot{x}_1 + 3kx_1 + k(x_1 - x_2) = 0 \\ &2m\ddot{x}_2 - k(x_1 - x_2) + k(x_2 - x_3) = 0 \\ &m\ddot{x}_3 - k(x_2 - x_3) = 0 \end{aligned}\right] \quad -(1)$$

further simplifying it we have

$$\left.\begin{aligned} &4m\ddot{x}_1 + 4kx_1 - kx_2 = 0 \\ &2m\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = 0 \\ &m\ddot{x}_3 - kx_2 + kx_3 = 0 \end{aligned}\right] \quad -(2)$$

In matrix form

$$\begin{bmatrix} 4m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 4K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad -13)$$

Now flexibility matrix $[A] = [K]^{-1}$

~~$[K]^{-1} = \frac{Adj[K]}{|K|}$~~

~~$Adj[K] =$~~

$$\text{so } [A] = \begin{bmatrix} 1/3K & 1/3K & 1/3K \\ 1/3K & 4/3K & 4/3K \\ 1/3K & 4/3K & 7/3K \end{bmatrix} \quad -14)$$

so the equation can be expressed as

$$[A][M]\{\ddot{x}\} + \{x\} = \{0\} \quad -15)$$

Assuming a steady state solution

$$\left.\begin{aligned} \{x\} &= \{X\} \sin\omega t \\ \{\dot{x}\} &= \omega\{X\}\cos\omega t \\ \{\ddot{x}\} &= -\omega^2\{X\}\sin\omega t \end{aligned}\right\} \quad -(6)$$

substituting the values in eq. 15)

$$\omega^2 \begin{bmatrix} \frac{1}{3K} & \frac{1}{3K} & \frac{1}{3K} \\ \frac{1}{3K} & \frac{4}{3K} & \frac{4}{3K} \\ \frac{1}{3K} & \frac{4}{3K} & \frac{7}{3K} \end{bmatrix} \begin{bmatrix} 4m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

$$\text{or } \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \quad -(7)$$

$$\text{Where } [B] = [A][M] = \frac{m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \quad -(8)$$

The iterative process for eq. (7) can be started by assuming a simple deflection shape.

1st Iteration

Let $x_1 = 1 \quad x_2 = 2 \quad x_3 = 3$

$$\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{Bmatrix} 11 \\ 32 \\ 41 \end{Bmatrix} = \frac{\omega^2 m}{3K} \times 11 \begin{Bmatrix} 1 \\ 2.9 \\ 3.7 \end{Bmatrix}$$

second iteration

Let $x_1 = 1 \quad x_2 = 2.9 \quad x_3 = 3.7$

$$\begin{Bmatrix} 1 \\ 2.9 \\ 3.7 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 2.9 \\ 3.7 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{Bmatrix} 13.5 \\ 42 \\ 53.1 \end{Bmatrix} = \frac{\omega^2 m}{3K} 13.5 \begin{Bmatrix} 1 \\ 3.1 \\ 3.9 \end{Bmatrix}$$

Third iteration

Let $x_1 = 1, \; x_2 = 3.1, \; x_2 = 3.9$

$$\begin{Bmatrix} 1 \\ 3.1 \\ 3.9 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 3.1 \\ 3.9 \end{Bmatrix} = \frac{\omega^2 m}{3K} 14.1 \begin{Bmatrix} 1 \\ 3.15 \\ 3.98 \end{Bmatrix}$$

fourth iteration

Let $x_1 = 1, \; x_2 = 3.15, \; x_2 = 3.98$

$$\begin{Bmatrix} 1 \\ 3.15 \\ 3.98 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 3.15 \\ 3.98 \end{Bmatrix} = \frac{\omega^2 m}{3K} 14.28 \begin{Bmatrix} 1 \\ 3.16 \\ 4.00 \end{Bmatrix}$$

It can be seen that the modes of 3rd and 4th iterations are repeatative with sufficient accuracy

therefore $\frac{\omega^2 m}{3K} \times 14.28 = 1$

$$\Rightarrow \omega = \sqrt{\frac{3}{14.28} \frac{k}{m}} = 0.458 \sqrt{k/m}$$

and the mode shapes are $(1, 3.16, 4.00)$

~~Higher Mode calculation:-~~

Orthogonality Principle:-

The matrix form of equation of motion for a n dof system can be expressed as:

$[m]\{\ddot{x}\} + [K]\{x\} = 0$ —(1)

Assuming a harmonic motion of frequency ω

$\{x\} = \{X\}\sin\omega t$

$\{\ddot{x}\} = -\omega^2\{X\}\sin\omega t$ —(2)

Substituting eq.(2) in eq.(1) we have

$-[m]\omega^2\{X\} + [K]\{X\} = 0$

or $[m]\omega^2\{X\} = [K]\{X\}$ —(3)

Now let r and s be two different modes of vibration and $\{X\}_r$ be the column giving amplitude of various rth mode and ω_r be the natural frequency of rth mode.

For the rth mode eq.(3) may be written as

$[m]\omega_r^2\{X\}_r = [K]\{X\}_r$ —(4)

Similarly for sth mode

$[m]\omega_s^2\{X\}_s = [K]\{X\}_s$ —(5)

Multiplying eq.(4) by the transpose of sth mode i.e $\{X\}'_s$ and eq.(5) by $\{X\}'_r$ we have,

$\omega_r^2\{X\}'_s[m]\{X\}_r = \{X\}'_s[K]\{X\}_r$ —(6)

$\omega_s^2\{X\}'_r[m]\{X\}_s = \{X\}'_r[K]\{X\}_s$ —(7)

Since $[m]$ and $[K]$ are symmetric matrices, we have

$$\left.\begin{aligned}\{X\}'_s[K]\{X\}_r &= \{X\}'_r[K]\{X\}_s\\ \{X\}'_s[m]\{X\}_r &= \{X\}'_r[m]\{X\}_s\end{aligned}\right\} \quad —(8)$$

Substituting eq.(8) in eq.(6) and eq.(7) and substracting eq.(7)

from eq. (6) we have

$$(\omega_r^2 - \omega_s^2)\{X\}'_r [m] \{X\}_s = 0 \quad \text{--- (9)}$$

Since r and s are two different modes, so $\omega_r \neq \omega_s$

So we have

$$\{X\}'_r [m] \{X\}_s = 0 \quad \text{--- (10)} \qquad \text{for } (r \neq s)$$

Eq. (10) may be expressed in a generalized form

$$\boxed{\sum_{i=1}^{n} m_i X_{ir} X_{is} = 0} \qquad r \neq s \quad \text{--- (11)}$$

This is called orthogonality principle.

Example

for a 3 dof system, the orthogonality principle may be written as:

$$\left.\begin{aligned} m_1 X_{11} X_{12} + m_2 X_{21} X_{22} + m_3 X_{31} X_{32} &= 0 \\ m_1 X_{11} X_{13} + m_2 X_{21} X_{23} + m_3 X_{31} X_{33} &= 0 \\ m_1 X_{12} X_{13} + m_2 X_{22} X_{23} + m_3 X_{32} X_{33} &= 0. \end{aligned}\right\} \text{--- (12)}$$

Calculation of Higher mode (using Sweeping Matrix method)

To find the second mode shape, applying orthogonality principle

$$m_1 X_{11} X_{12} + m_2 X_{21} X_{22} + m_3 X_{31} X_{32} = 0.$$

or $$X_{12} = -\frac{m_2}{m_1}\left(\frac{X_{21}}{X_{11}}\right) X_{22} - \left(\frac{m_3}{m_1}\right)\left(\frac{X_{31}}{X_{11}}\right) X_{32}$$

in matrix form

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2 = \begin{bmatrix} 0 & -\frac{m_2}{m_1}\left(\frac{X_2}{X_1}\right)_1 & -\frac{m_3}{m_1}\left(\frac{X_3}{X_1}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}_2$$

or $X = SX$

where S = sweeping matrix

In matrix form

$$[S_2] = \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix}$$

so the equation in matrix form.

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & -0.79 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

or $$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 0 & 0 & 0.42 \\ 0 & 0 & -1.32 \\ 0 & 0 & 1.68 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

starting with a assummed value. $\{1 \ 0 \ 1\}'$

$$\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 0 & 0 & 0.42 \\ 0 & 0 & -1.32 \\ 0 & 0 & 1.68 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{Bmatrix} 0.42 \\ -1.32 \\ 1.68 \end{Bmatrix}$$

$$= \frac{0.42\,\omega^2 m}{3K} \begin{Bmatrix} 1 \\ -3.14 \\ 4 \end{Bmatrix}$$

Now taking $x_1 = 1 \quad x_2 = -3.14 \quad x_3 = 4$

$$\begin{Bmatrix} 1 \\ -3.14 \\ 4 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 0 & 0 & 0.42 \\ 0 & 0 & -1.32 \\ 0 & 0 & 1.68 \end{bmatrix} \begin{Bmatrix} 1 \\ -3.14 \\ 4 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{Bmatrix} 1.68 \\ -5.28 \\ 6.72 \end{Bmatrix}$$

$$= \frac{1.68\,\omega^2 m}{3K} \begin{Bmatrix} 1 \\ -3.14 \\ 4 \end{Bmatrix}$$

As the modes are repeatative

$$\frac{1.68\,\omega^2 m}{3K} = 1 \implies \omega = \sqrt{\frac{3}{1.68}\frac{k}{m}} = 1.336\sqrt{\frac{k}{m}}$$

The final result may be summarized as

$$\boxed{\begin{aligned} \omega_1 &= 0.455\sqrt{\frac{k}{m}} \\ \omega_2 &= \sqrt{k/m} \\ \omega_3 &= 1.336\sqrt{\frac{k}{m}} \end{aligned}}$$

$$S = \begin{bmatrix} 0 & -\frac{m_2}{m_1}\left(\frac{x_2}{x_1}\right) & -\frac{m_3}{m_1}\left(\frac{x_3}{x_1}\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the new equation for second mode

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 4 & 2 & 1 \\ 4 & 8 & 4 \\ 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 0 & -1.58 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

or

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Taking a trial value of $\{1 0 -1\}'$, ~~and after~~

$$\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 0 & -4.32 & -3 \\ 0 & 1.68 & 0 \\ 0 & 1.68 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = \frac{\omega^2 m}{3K} \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$$

or

$$\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = \frac{\omega^2 m}{K} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

therefore $\frac{\omega^2 m}{K} = 1$ or $\omega = \sqrt{\frac{k}{m}}$

and the mode shape are $(1, 0, -1)$

Third mode :-

For the third mode applying orthogonality principle as!

$$m_1 x_{11} x_{13} + m_2 x_{21} x_{23} + m_3 x_{31} x_{33} = 0$$

and $m_1 x_{12} x_{13} + m_2 x_{22} x_{23} + m_3 x_{32} x_{33} = 0$.

or $4m\, x_{13} + 2m\,(3.16)\, x_{23} + m\,(4)\, x_{33} = 0$

and $4m\, x_{13} + m\,(-1)\, x_{33} = 0$.

on solving $\boxed{\begin{aligned} x_{13} &= 0.25\, x_{33} \\ x_{23} &= -0.79\, x_{33} \end{aligned}}$ and $x_{33} = x_{33}$

MODULE IV

TORSIONAL VIBRATION

Single Rotor System

If a rigid body oscillates about a specific reference axis, the resulting motion is called torsional vibration. In this case, the displacement of the body is measured in terms of an angular coordinate. In a torsional vibration problem, the restoring moment may be due to the torsion of an elastic member or to the unbalanced moment of a force or couple. Figure 1 shows a disc, which has a polar mass moment of inertia J_0 mounted at one end of a solid circular shaft, the other end of which is fixed. Let the angular rotation of the disc about the axis of the shaft be θ, θ also represents the shaft's angle of twist.

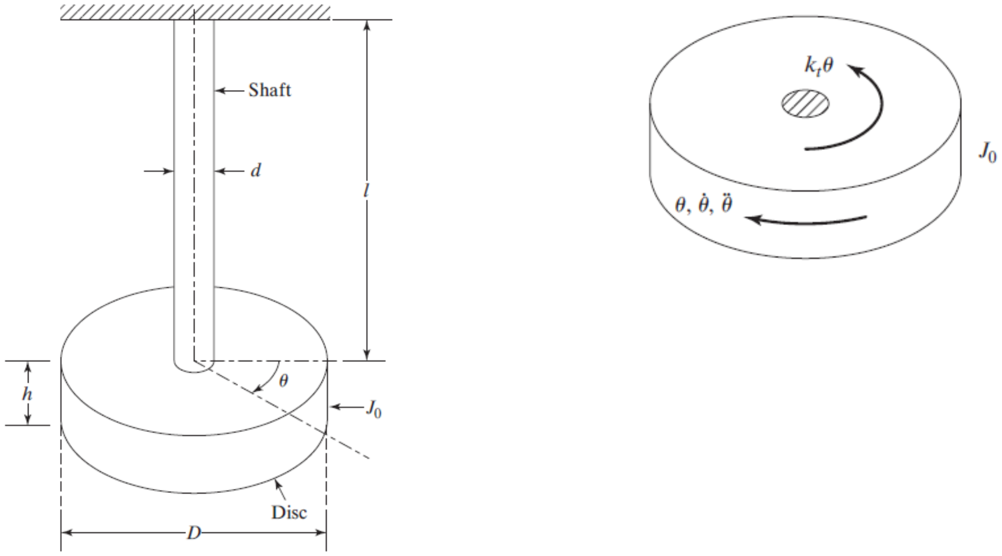


Figure 1 Torsional vibration of a disc

Let

θ = angular twist of the disc from its equilibrium position

T = torque required to produce the twist = $\dfrac{GJ}{l}\theta$

J is the polar moment inertia of the rod = $\pi d^4 / 32$

d = rod dia.

l = rod length

Then the torsional spring constant can be defined as,

$$\boxed{k_t = \frac{T}{\theta} = \frac{GJ}{l}}$$

Applying D'Alembert's principle the equation of motion may be written as

$$I\ddot{\theta} + k_t\theta = 0$$

$$\ddot{\theta} + \frac{k_t}{I}\theta = 0$$

So the natural frequency ω_n may written as

$$\omega_n = \sqrt{\frac{k_t}{I}}$$

And $\boxed{f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_t}{I}}}$ Hz

Double Rotor System

Consider a torsional system consisting of two discs mounted on a shaft, as shown in Fig. 2. The three segments of the shaft have rotational spring constants k_{t1}, k_{t2}, k_{t3} and as indicated in the figure. Also shown are the discs of mass moments of inertia J_1 and J_2 and the applied torques M_{t1} and $M_{t2}$and and the rotational degrees of freedom θ_1 and θ_2 and The differential equations of rotational motion J_1 and $J_2$for the discs and can be derived as:

$$J_1\ddot{\theta}_1 = -k_{t1}\theta_1 + k_{t2}(\theta_2 - \theta_1) + M_{t1}$$

$$J_2\ddot{\theta}_2 = -k_{t2}(\theta_2 - \theta_1) - k_{t3}\theta_2 + M_{t2}$$

which upon rearrangement become

$$J_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 = M_{t1}$$

$$J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2} + k_{t3})\theta_2 = M_{t2}$$

For the free-vibration analysis of the system, Eq. (5.19) reduces to

$$J_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 = 0$$

$$J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2} + k_{t3})\theta_2 = 0$$

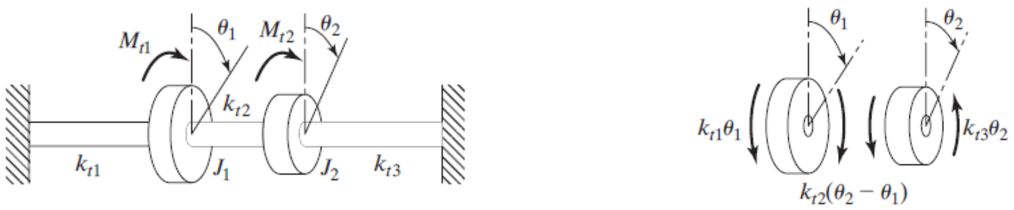


Figure 2 Torsional vibration of a two rotor system

Example

Find the natural frequencies and mode shapes for the torsional system shown in Fig. 5.9 for $J_1 = J_0$, $J_2 = 2J_0$, and $k_{t1} = k_{t2} = k_t$.

Solution: The differential equations of motion, reduce to (with $k_{t3} = 0$, $k_{t1} = k_{t2} = k_t$, $J_1 = J_0$, and $J_2 = 2J_0$):

$$J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = 0 \tag{E.1}$$

Rearranging and substituting the harmonic solution

$$\theta_i(t) = \Theta_i \cos(\omega t + \phi); \qquad i = 1, 2 \tag{E.2}$$

gives the frequency equation:

$$2\omega^4 J_0^2 - 5\omega^2 J_0 k_t + k_t^2 = 0 \tag{E.3}$$

The solution of Eq. (E.3) gives the natural frequencies

$$\omega_1 = \sqrt{\frac{k_t}{4J_0}(5 - \sqrt{17})} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{k_t}{4J_0}(5 + \sqrt{17})} \tag{E.4}$$

The amplitude ratios are given by

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = 2 - \frac{(5 - \sqrt{17})}{4}$$

Transverse vibration of beam with various boundary conditions

Consider the free-body diagram of an element of a beam shown in Fig. , where $M(x, t)$ is the bending moment, $V(x, t)$ is the shear force, and $f(x, t)$ is the external force per unit length of the beam. Since the inertia force acting on the element of the beam is

$$\rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

the force equation of motion in the z direction gives

$$-(V + dV) + f(x, t)\, dx + V = \rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

where ρ is the mass density and $A(x)$ is the cross-sectional area of the beam. The moment equation of motion about the y-axis passing through point O in Fig. leads to

$$(M + dM) - (V + dV)\, dx + f(x, t)\, dx \frac{dx}{2} - M = 0$$

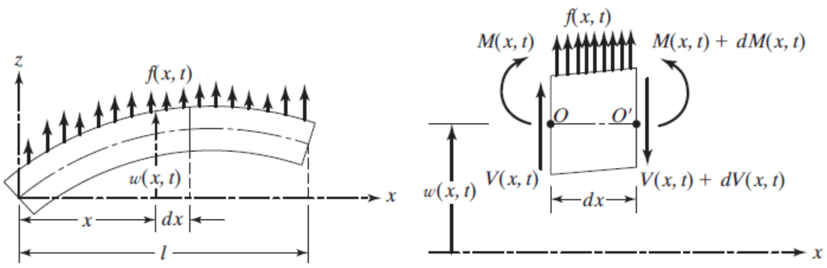


Figure 3 Transverse vibration of beam

By writing

$$dV = \frac{\partial V}{\partial x} dx \qquad \text{and} \qquad dM = \frac{\partial M}{\partial x} dx$$

and disregarding terms involving second powers in dx, Eqs. can be written as

$$-\frac{\partial V}{\partial x}(x, t) + f(x, t) = \rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t)$$

$$\frac{\partial M}{\partial x}(x, t) - V(x, t) = 0$$

By using the relation $V = \partial M / \partial x$ from Eq. becomes

$$-\frac{\partial^2 M}{\partial x^2}(x, t) + f(x, t) = \rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t)$$

From the elementary theory of bending of beams (also known as the *Euler-Bernoulli* or *thin beam theory*), the relationship between bending moment and deflection can be expressed as

$$M(x, t) = EI(x) \frac{\partial^2 w}{\partial x^2}(x, t)$$

where E is Young's modulus and $I(x)$ is the moment of inertia of the beam cross section about the y-axis. Inserting Eq. we obtain the equation of motion for the forced lateral vibration of a nonuniform beam:

$$\frac{\partial^2}{\partial x^2}\left[EI(x) \frac{\partial^2 w}{\partial x^2}(x, t)\right] + \rho A(x) \frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t)$$

For a uniform beam, Eq. reduces to

$$EI \frac{\partial^4 w}{\partial x^4}(x, t) + \rho A \frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t)$$

For free vibration, $f(x, t) = 0$, and so the equation of motion becomes

$$c^2 \frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0$$

where

$$c = \sqrt{\frac{EI}{\rho A}}$$

Since the equation of motion involves a second-order derivative with respect to time and a fourth-order derivative with respect to x, two initial conditions and four boundary conditions are needed for finding a unique solution for $w(x, t)$. Usually, the values of lateral displacement and velocity are specified as $w_0(x)$ and $\dot{w}_0(x)$ at $t = 0$, so that the initial conditions become

$$w(x, t = 0) = w_0(x)$$

$$\frac{\partial w}{\partial t}(x, t = 0) = \dot{w}_0(x)$$

The free-vibration solution can be found using the method of separation of variables as

$$w(x, t) = W(x)T(t)$$

Substituting and rearranging leads to

$$\frac{c^2}{W(x)} \frac{d^4W(x)}{dx^4} = -\frac{1}{T(t)} \frac{d^2T(t)}{dt^2} = a = \omega^2$$

where $a = \omega^2$ is a positive constant . Equation can be written as two equations:

$$\frac{d^4W(x)}{dx^4} - \beta^4 W(x) = 0$$

$$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0$$

where

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI}$$

The solution of Eq. can be expressed as

$$T(t) = A \cos \omega t + B \sin \omega t$$

where A and B are constants that can be found from the initial conditions. For the solution of Eq. , we assume

$$W(x) = Ce^{sx}$$

where C and s are constants, and derive the auxiliary equation as

$$s^4 - \beta^4 = 0$$

The roots of this equation are

$$s_{1,2} = \pm\beta, \qquad s_{3,4} = \pm i\beta$$

Hence the solution of Eq. becomes

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

where C_1, C_2, C_3, and C_4 are constants. Equation can also be expressed as

$$W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x$$

or

$$W(x) = C_1(\cos \beta x + \cosh \beta x) + C_2(\cos \beta x - \cosh \beta x) + C_3(\sin \beta x + \sinh \beta x) + C_4(\sin \beta x - \sinh \beta x)$$

where C_1, C_2, C_3, and C_4, in each case, are different constants. The constants C_1, C_2, C_3, and C_4 can be found from the boundary conditions. The natural frequencies of the beam are computed from Eq. as

$$\omega = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta l)^2 \sqrt{\frac{EI}{\rho A l^4}}$$

The function $W(x)$ is known as the *normal mode* or *characteristic function* of the beam and ω is called the *natural frequency of vibration*. For any beam, there will be an infinite number of normal modes with one natural frequency associated with each normal mode. The unknown constants C_1 to C_4 in Eq. and the value of β in Eq. can be determined from the boundary conditions of the beam as indicated below.

The common boundary conditions are as follows:

1. *Free end:*

$$\text{Bending moment} = EI\frac{\partial^2 w}{\partial x^2} = 0$$

$$\text{Shear force} = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = 0$$

2. *Simply supported (pinned) end:*

$$\text{Deflection} = w = 0, \qquad \text{Bending moment} = EI\frac{\partial^2 w}{\partial x^2} = 0$$

3. *Fixed (clamped) end:*

$$\text{Deflection} = 0, \qquad \text{Slope} = \frac{\partial w}{\partial x} = 0$$

The frequency equations, the mode shapes (normal functions), and the natural frequencies for beams with common boundary conditions are given in Fig.
We shall now consider some other possible boundary conditions for a beam.

4. *End connected to a linear spring, damper, and mass* : When the end of a beam undergoes a transverse displacement w and slope $\partial w/\partial x$. with velocity $\partial w/\partial t$ and acceleration $\partial^2 w/\partial t^2$, the resisting forces due to the spring, damper, and mass are proportional to w, $\partial w/\partial t$, and $\partial^2 w/\partial t^2$, respectively. This resisting force is balanced by the shear force at the end. Thus

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = a\left[kw + c\frac{\partial w}{\partial t} + m\frac{\partial^2 w}{\partial t^2}\right]$$

where $a = -1$ for the left end and $+1$ for the right end of the beam. In addition, the bending moment must be zero; hence

$$EI\frac{\partial^2 w}{\partial x^2} = 0$$

5. *End connected to a torsional spring, torsional damper, and rotational inertia* (Fig. 8.16(b)): In this case, the boundary conditions are

$$EI\frac{\partial^2 w}{\partial x^2} = a\left[k_t\frac{\partial w}{\partial x} + c_t\frac{\partial^2 w}{\partial x\partial t} + I_0\frac{\partial^3 w}{\partial x\partial t^2}\right]$$

where $a = +1$ for the left end and -1 for the right end of the beam, and

Commonly used boundary conditions for the transverse vibration of beam are as shown in Figure 4

Pinned-pinned	$\sin\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x]$	$\beta_1 l = \pi$ $\beta_2 l = 2\pi$ $\beta_3 l = 3\pi$ $\beta_4 l = 4\pi$
Free-free	$\cos\beta_n l \cdot \cosh\beta_n l = 1$	$W_n(x) = C_n[\sin\beta_n x + \sinh\beta_n x + \alpha_n(\cos\beta_n x + \cosh\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l - \sinh\beta_n l}{\cosh\beta_n l - \cos\beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$ ($\beta l = 0$ for rigid-body mode)
Fixed-fixed	$\cos\beta_n l \cdot \cosh\beta_n l = 1$	$W_n(x) = C_n[\sinh\beta_n x_n x - \sin\beta_n x + \alpha_n(\cosh\beta_n x - \cos\beta_n x)]$ where $\alpha_n = \left(\frac{\sinh\beta_n l - \sin\beta_n l}{\cos\beta_n l - \cosh\beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$
Fixed-free	$\cos\beta_n l \cdot \cosh\beta_n l = -1$	$W_n(x) = C_n[\sin\beta_n x - \sinh\beta_n x - \alpha_n(\cos\beta_n x - \cosh\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l + \sinh\beta_n l}{\cos\beta_n l + \cosh\beta_n l}\right)$	$\beta_1 l = 1.875104$ $\beta_2 l = 4.694091$ $\beta_3 l = 7.854757$ $\beta_4 l = 10.995541$
Fixed-pinned	$\tan\beta_n l - \tanh\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x - \sinh\beta_n x + \alpha_n(\cosh\beta_n x - \cos\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l - \sinh\beta_n l}{\cos\beta_n l - \cosh\beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$
Pinned-free	$\tan\beta_n l - \tanh\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x + \alpha_n\sinh\beta_n x]$ where $\alpha_n = \left(\frac{\sin\beta_n l}{\sinh\beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$ ($\beta l = 0$ for rigid-body mode)

Figure 4 Commonly used boundary conditions for transverse vibration of beam

References

1. Mechanical Vibration by Morse and Hinkle
2. Mechanical Vibration with application by W.T. Thomas
3. Mechanical Vibrations by V.P. Singh
4. Mechanical Vibrations by S.S. RAo
5. Mechanical Vibrations by G.K. Grover