B.Tech 4th Semester MATHEMATICS-IV

UNIT-1

NUMERICAL METHOD

We use numerical method to find approximate solution of problems by numerical calculations with aid of calculator. For better accuracy we have to minimize the error.

Error = Exact value – Approximate value

Absolute error = modulus of error

Relative error = Absolute error / (Exact value)

Percentage error = 100 X Relative error

The error obtained due to rounding or chopping is called rounding error.

For example $\pi = 3.14159$ is approximated as 3.141 for chopping (deleting all decimal)

or 3.142 for rounding up to 3 decimal places.

Significant digit:

It is defined as the digits to the left of the first non-zero digit to fix the position of decimal point.

For example each of following numbers has 5 significant digits.

0.00025610, 25.610, 25601, 25610

Solution of Equations by Iteration:

Intermediate value Theorem: If a function f(x) is continuous in closed interval [a,b] and satisfies $f(a)f(b) < 0$ then there exists atleast one real root of the equation $f(x) = 0$ in open interval (a,b).

Algebraic equations are equations containing algebraic terms (different powers of x). For example $x^2-7x+6=0$

Transcendental equations are equations containing non-algebraic terms like trigonometric, exponential, logarithmic terms. For example $\sin x - e^x = 0$

A. Fixed point iteration method for solving equation f(x) = 0
Procedure

Step-I We rewrite the equation $f(x) = 0$ of the form $x = h(x)$, $x=g(x)$, $x = D(x)$

We find the interval (a,b) containing the solution (called root).

Step-II We choose that form say $x = h(x)$ which satisfies $|h'(x)| < 1$ in interval (a,b) containing the solution (called root).

Step-III We take $x_{n+1} = h(x_n)$ as the successive formula to find approximate solution (root) of the equation $f(x) = 0$

Step-III Let $x=x_0$ be initial guess or initial approximation to the equation $f(x) = 0$

Then x_1=h(x_1) , x_2=h(x_2) , x_3=h(x_3) and so on.We will continue this process till we get solution (root) of the equation f(x) = 0 up to desired accuracy.

Convergence condition for Fixed point iteration method

If x=a is a root of the equation f(x) = 0 and the root is in interval (a, b). The function h′(x) and h(x) defined by x = h(x) Is continuous in (a,b) .Then the approximations x_1=h(x_1) , x_2=h(x_2) , x_3=h(x_3) .... converges to the root x=a provided **|** h′(x) **|** < 1 in interval (a,b) containing the root for all values of x.

Problems

1. **Solve x^3 - sin x -1 =0 correct to two significant figures by fixed point iteration method correct up to 2 decimal places.**

Solution: x^3 - sin x -1 =0........ ....(1)

Let f(x) = x^3-sin x -1

f(0) = -1, f(1)= - 0.8415, f(2)=6.0907

As f(1)f(2)< 0 by Intermediate value Theorem the root of real root of the equation f(x) = 0 lies between 1 and 2

Let us rewrite the equation f(x) = 0 of the form x = h(x)

x= $(1 + \text{Sin } x)^{1/3}$ = $h_1(x)$ and x = $\text{Sin}^{-1}(x^3 - 1)$= $h_2(x)$

We see that **|** $h_1'(x)$ **|** < 1 in interval (1,2) containing the root for all values of x.

We use x_{n+1}= $(1 + \text{Sin } x_n)^{1/3}$ as the successive formula to find approximate solution (root) of the equation (1).

Let x_0 =1.5 be initial guess to the equation (1).

Then x_1= $(1 + \text{Sin } x_0)^{1/3}$ = $(1 + \text{Sin } 1.5)^{1/3}$ = 1.963154

x_2= $(1 + \text{Sin } x_1)^{1/3}$ = $(1 + \text{Sin } 1.963154)^{1/3}$ = 1.460827

x_3= $(1 + \text{Sin } x_2)^{1/3}$ = $(1 + \text{Sin } 1.460827)^{1/3}$ = 1.440751

x_4= $(1 + \text{Sin } x_3)^{1/3}$ = $(1 + \text{Sin } 1.440751)^{1/3}$ = 1.441289

which is the root of equation (1) correct to two decimal places.

Newton Raphson Method

Procedure

Step-I We find the interval (a,b) containing the solution (called root) of the equation f(x) = 0 .

Step-II Let x=x_0 be initial guess or initial approximation to the equation f(x) = 0

Step-III We use $x_{n+1} = x_n - [f(x_n) / f'(x_n)]$ as the successive formula to find approximate solution (root) of the equation $f(x) = 0$

Step-III Then x_1, x_2, x_3 .... and so on are calculated and we will continue this process till we get root of the equation $f(x) = 0$ up to desired accuracy.

2. Solve x - 2sin x - 3 = 0 correct to two significant figures by Newton Raphson method correct up to 5 significant digits.

Solution: $x - 2\sin x - 3 = 0$........ ....(2)

Let $f(x) = x - 2\sin x - 3$

$f(0) = -3$, $f(1) = -2 - 2\,\mathrm{Sin}\,1$, $f(2) = -1 - 2\,\mathrm{Sin}\,2$, $f(3) = -2\,\mathrm{Sin}\,3$, $f(4) = 1 - 2\,\mathrm{Sin}\,4$

$f(-2) = -5 + 2\,\mathrm{Sin}\,2$, $f(-1) = -4 + 2\sin 1$

As $f(3)f(4) < 0$ by Intermediate value Theorem the root of real root of the equation $f(x) = 0$ lies between 3 and 4

Let Let $x_0 = 4$ be the initial guess to the equation (2).

Then $x_1 = x_0 - [f(x_0) / f'(x_0)] = 2 - f(2)/ f'(2) = 3.09900$

$x_2 = x_1 - [f(x_1) / f'(x_1)] = - 1.099 - f(- 1.099)/ f'(- 1.099) = 3.10448$

$x_3 = x_2 - [f(x_2) / f'(x_2)] = 3.10450$

$x_4 = x_3 - [f(x_3) / f'(x_3)] = 3.10451$

which is the root of equation (2) correct to five significant digits.

Secant Method

Procedure

Step-I We find the interval (a,b) containing the solution (called root) of the equation $f(x) = 0$.

Step-II Let $x = x_0$ be initial guess or initial approximation to the equation $f(x) = 0$

Step-III We use $x_{n+1} = x_n - [(x_n - x_{n-1})f(x_n)] / [f(x_n) - f(x_{n-1})]$ as the successive formula to find approximate solution (root) of the equation $f(x) = 0$

Step-III Then x_1, x_2, x_3 .... and so on are calculated and we will continue this process till we get root of the equation $f(x) = 0$ up to desired accuracy.

3 . Solve Cos x = x e^x correct to two significant figures by Secant method correct up to 2 decimal places.

Solution: $\mathrm{Cos}\,x = x e^x$ ....(3)

Let $f(x) = \mathrm{Cos}\,x - x e^x$

f(0) = 1, f(1)= Cos 1 – e = - 2 .178

As f(0)f(1)< 0 by Intermediate value Theorem the root of real root of the equation f(x) = 0 lies between 0 and 1

Let Let $x_0 = 0$ and $x_1 = 1$ be two initial guesses to the equation (3).

Then

$$x_2 = x_1 - \frac{(x_1 - x_0) f(x_1)}{f(x_1) - f(x_0)} = 1 - \frac{(1-0) f(1)}{f(1) - f(0)} = 1 - \frac{2.178}{-3.178} = 0.31465$$

$f(x_2)$= f (0.31465)= Cos (0.31465) - 0.31465 $e^{0.31465}$ = 0.51987

$$x_3 = x_2 - \frac{(x_2 - x_1) f(x_2)}{f(x_2) - f(x_1)} = 0.31465 - \frac{(0.31465 - 1) f(0.31465)}{f(0.31465) - f(1)} = 0.44672$$

$$x_4 = x_3 - \frac{(x_3 - x_2) f(x_3)}{f(x_3) - f(x_2)} = 0.64748$$

$$x_5 = x_4 - \frac{(x_4 - x_3) f(x_4)}{f(x_4) - f(x_3)} = 0.44545$$

which is the root of equation (3) correct to two decimal places.

4. Solve x^4 - x - 7 = 0 correct to two significant figures by Newton- Raphson method correct up to 6 significant digits.

Solution: x^4 - x - 7 = 0........ ....(4)

Let f(x) = x^4 - x - 7

f(0) = -7, f(1)= -7 , f(2)= 5

As f(1)f(2)< 0 by Intermediate value Theorem the root of real root of the equation f(x) = 0 lies between 1 and 2

Let Let x_0 = 1.5 be the initial guess to the equation (2).

Then x_1= x_0 - [f(x_0) / f′(x_0)] = 1.5 - f(1.5)/ f′(1.5) = 1.78541

x_2= x_1 - [f(x_1) / f′(x_1)] = 1.7854- f1.7854)/ f′(1.7854) = 1.85876

x_3= x_2 - [f(x_2) / f′(x_2)] = 1.85643

x_4= x_3 - [f(x_3) / f′(x_3)] = 1.85632

which is the root of equation (2) correct to 6S.

INTERPOLATION

Interpolation is the method of finding value of the dependent variable y at any point x using the following given data.

x	x_0	x_1	x_2	x_3	..	..	..	x_n
y	y_0	y_1	y_2	y_3	..	..	..	y_n

This means that for the function y = f(x) the known values at x = $x_0, x_1, x_2, \ldots\ldots\ldots, x_n$ are respectively

y = $y_0, y_1, y_2, \ldots\ldots\ldots, y_n$ and we want to find value of y at any point x.

For this purpose we fit a polynomial to these datas called interpolating polynomial. After getting the polynomial p(x) which is an approximation to f(x), we can find the value of y at any point x.

Finite difference operators

Let us take equispaced points $x_0, x_1, x_2, \ldots\ldots\ldots, x_n$

i.e. $x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

Forward difference operator $\Delta y_n = y_{n+1} - y_n$

Backward difference operator $\nabla y_n = y_n - y_{n-1}$

Central difference operator $\delta y_i = y_{i+1/2} - y_{i-½}$

Shift Operator $E y_i = y_{i+1}$

Newton's Forward difference Interpolation formula

Let us take the equi-spaced points x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

Then $\Delta y_n = y_{n+1} - y_n$ is called the first Forward difference

i.e. $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$ and so on.

$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$ is called the second Forward difference

i.e. $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$, $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ and so on.

Newton's Forward difference Interpolation formula is

$$Pn(x) = y_0 + p\Delta y_0 + [p(p-1)/2!]\Delta^2 y_0 + [p(p-1)(p-2)/3!]\Delta^3 y_0$$
$$+ \ldots\ldots\ldots\ldots + [p(p-1)(p-2)\ldots\ldots.(p\text{-}n\text{-}1)/n!]\Delta^n y_0$$

Where $p = (x - x_0)/h$

Problems

5. **Using following data find the Newton's interpolating polynomial and also find the value of y at x=5**

x	0	10	20	30	40
y	7	18	32	48	85

Solution

Here $x_0 = 0, x_1 = 10, x_2 = 20, x_3 = 30, x_4 = 40,$

$x_1 - x_0 = 10 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$

The given data is equispaced.

As x= 5 lies between 0 and 10 and at the start of the table and data is equispaced, we have to use Newton's forward difference Interpolation.

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	**7**				
		11			
10	18		**03**		
		14		**02**	
20	32		05		**10**
		19		12	
30	51		17		
		36			
40	87				

Here $x_0 = 0$, $y_0 = 7$, $h = x_1 - x_0 = 10-0 = 10$

$\Delta y_0 = 11$, $\Delta^2 y_0 = 3$,

$\Delta^3 y_0 = 2$, $\Delta^4 y_0 = 10$

$p = (x - x_0)/h = (x - 0)/10 = 0.1x$

$P_n(x) = y_0 + p\Delta y_0 + [\,p(p-1)/2!\,]\,\Delta^2 y_0 + [\,p(p-1)(p-2)/3!\,]\,\Delta^3 y_0$

$+ [\,p(p-1)(p-2)(p-3)/4!\,]\,\Delta^4 y_0$

$= 7 + 0.1x\,(11) + [0.1x(0.1x - 1)/2!\,]\,(3) + [0.1x(0.1x - 1)\,(0.1x - 2)/3!\,]\,(2)$

$+ [0.1x(0.1x - 1)\,(0.1x - 2)\,(0.1x - 3)/4!\,]\,(10)$

$= 7 + 1.1x + (0.01x^2 - 0.1x)1.5 + (0.001x^3 - 0.03x^2 + 0.2x)/3$

$+ 0.416\,(0.0001x^4 - 0.006x^3 + 0.11x^2 - 0.6x)$

$P_n(x) = 0.0000416\,x^4 - 0.0022\,x^3 + 0.05x^2 + 1.26\,x + 7$

Is the **Newton's** interpolating polynomial

To find the approximate value of y at x=5 we put x=5 in the interpolating polynomial to get

$y(5) = P_n(5) = 0.0000416\,(5)^4 - 0.0022\,(5)^3 + 0.05(5)^2 + 1.26\,(5) + 7 = 14.301$

6. **Using following data find the Newton's interpolating polynomial and also find the value of y at x=24**

x	20	35	50	65	80
y	3	11	24	50	98

Solution

Here $x_0 = 20$, $x_1 = 35$, $x_2 = 50$, $x_3 = 65$, $x_4 = 80$,

$x_1 - x_0 = 15 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$

The given data is equispaced.

As x= 24 lies between 20 and 35 and at the start of the table and data is equispaced, we have to use Newton's forward difference Interpolation.

Here $x_0 = 20$, $y_0 = 3$, $h = x_1 - x_0 = 35 - 20 = 15$

$\Delta y_0 = 8$, $\Delta^2 y_0 = 5$,

$\Delta^3 y_0 = 8$, $\Delta^4 y_0 = 1$

$p = (x - x_0)/h = (x - 20)/15 = 0.0666\,x - 1.333333$

Forward difference table

x	y	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
20	**3**				
		8			
35	11		**05**		
		13		**08**	
50	24		13		0**1**
		26		9	
65	50		22		
		48			
80	98				

$P_n(x) = y_0 + p\,\Delta y_0 + [p(p-1)/2!]\,\Delta^2 y_0 + [p(p-1)(p-2)/3!]\,\Delta^3 y_0$

$+ [p(p-1)(p-2)(p-3)/4!]\,\Delta^4 y_0$

$= 3 + 8(0.0666x - 1.333333) + 5[(0.0666x - 1.333333)(0.0666x - 2.333333)/2!]$

$+ 8[(0.0666x - 1.333333)(0.0666x - 2.333333)(0.0666x - 3.333333)/3!]$

$+ [(0.0666x - 1.333333)(0.0666x - 2.333333)(0.0666x - 3.333333)(0.0666x - 4.333333)/4!]$

$= 3 + 0.53333333x - 10.666666 + 0.01111x^2 - 0.16666666x + 7.777777$

$+ [(0.5333333x - 10.66666)(0.0666x - 2.333333)(0.011111x - 0.5555555)]$

$+ [(0.0666x - 1.333333)(0.0666x - 2.333333)(0.011111x - 0.5555555)(0.01666x - 1.083333)]$

Is the **Newton's** interpolating polynomial

To find the approximate value of y at x = 24 we put x = 24 in the interpolating polynomial to get

$y(24) = P_n(24) = 3 + (0.53333333)24 - 10.666666 + 0.01111(24^2) - (0.16666666)24 + 7.777777$

$+ [(0.5333333(24) - 10.66666)(0.0666(24) - 2.333333)(0.011111(24) - 0.5555555)]$

$+ [(1.59999 - 1.333333)(1.59999 - 2.333333)(0.266666 - 0.5555555)(0.399999 - 1.083333)]$

Newton's Backward difference Interpolation formula

Let us take the equi-spaced points x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, ..., $x_n = x_{n-1} + h$

Then $\nabla y_n = y_n - y_{n-1}$ is called the first backward difference

i.e. $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$ and so on.

$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$ is called the second backward difference

i.e. $\nabla^2 y_1 = \nabla y_1 - \nabla y_0$, $\nabla^2 y_2 = \nabla y_2 - \nabla y_1$ and so on.

Newton's backward difference Interpolation formula is

$Pn(x) = y_n + p \nabla y_n + [p(p+1)/2!] \nabla^2 y_n + [p(p+1)(p+2)/3!] \nabla^3 y_n$

$+ \ldots\ldots\ldots\ldots + [p(p+1)(p+2)\ldots\ldots.(p+n-1)/n!] \nabla^n y_n$

Where $p = (x - x_n)/h$

7. Using following data to find the value of y at x = 35

x	0	10	20	30	40
y	7	18	32	48	85

Solution :

Here $x_0 = 0$, $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, $x_4 = 40$,

$x_1 - x_0 = 10 = x_2 - x_1 = x_3 - x_2 = x_4 - x_3$

The given data is equispaced.

As x= 35 lies between 3 0 and 40 and at the end of the table and given data is equispaced ,we have to use Newton's Backward difference Interpolation.

Here $x = 35$, $x_n = 40$, $y_n = 87$, $h = x_1 - x_0 = 10-0 = 10$

$\nabla y_n = 36$, $\nabla^2 y_n = 17$,

$\nabla^3 y_n = 12$, $\nabla^4 y_n = 10$

$p = (x - x_n)/h = (35 - 40)/10 = -0.5$

Backward difference table

x	y	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
0	7				
		11			
10	18		03		
		14		02	
20	32		05		**10**
		19		**12**	
30	51		**17**		
		36			
40	**87**				

$P_n(x) = y_n + p \nabla y_n + [\, p(p+1)/2! \,] \nabla^2 y_n + [\, p(p+1)(p+2)/3! \,] \nabla^3 y_n$

$+ [\, p(p+1)(p+2)(p+3)/4! \,] \nabla^4 y_n$

$= 87 + (-0.5)(36) + (-0.5)(-0.5+1)(17)/2! + (-0.5)(-0.5+1)(-0.5+2)(12)/3!$

$+ (-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(10)/4!$

$= 87 - 18 - 0.25(8.5) - 0.25(18)/6 - 0.25(15)(2.5)/24$

$= 65.734375$

This is the approximate value of y at x=35

$y(35)=P_n(35) = 65.734375$

Inverse Interpolation

The process of finding the independent variable x for given values of f(x) is called Inverse Interpolation .

8. Solve ln x = 1.3 by inverse Interpolation using x= G(y) with G(1)=2.718 ,G(1.5)= 4.481 , G(2)= 7.387 ,G(2.5)= 12.179 and find value of x

Forward difference table

y	x	Δ y	Δ^2 y	Δ^3 y
1	**2.718**			
		1.763		
1.5	4.481		**1.143**	
		2.906		**0.743**
2	7.387		1.886	
		4.792		
2.5	12.179			

Here $y_0 = 1$, $h=y_1 - y_0 = 1.5 - 1 = 0.5$

$x_0 = 2.718$, $\Delta x_0 = 1.763$, $\Delta^2 x_0 = 1.143$,

$\Delta^3 x_0 = 0.743$

$p = (y - y_0)/h = (1.3 - 1)/0.5 = 0.6$

Newton's Forward difference Interpolation formula is

$Pn (y) = x_0 + p\,\Delta x_0 + [\, p(p-1)/2!\,]\, \Delta^2 x_0 + [\, p(p-1)(p-2)/3!\,]\, \Delta^3 x_0$

$= 2.718 + 0.6 (1.763)+ 0.6(0.6-1)1.143/2 + 0.6(0.6-1) (0.6-2)0.743/6$

$= 3.680248$

Lagrange Interpolation (data may not be equispaced)

Lagrange Interpolation can be applied to arbitrary spaced data.

Linear interpolation is interpolation by the line through points (x_1,y_1) and (x_0,y_0)

Linear interpolation is $P_1(x)= l_0 y_0 + l_1 y_1$

Where $l_0 = (x- x_1) /(x_0- x_1)$ and $l_1 = (x- x_0) /(x_1- x_0)$

Quadratic Lagrange Interpolation is the Interpolation through three given points (x_2,y_2) , (x_1,y_1) and (x_0,y_0) given by the formula

$P_2(x)= l_0\, y_0 + l_1\, y_1 + l_2\, y_2$

Where $l_0 = \dfrac{(x-x_2)(x-x_1)}{(x_0-x_2)(x_0-x_1)}$, $l_1 = \dfrac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)}$ $and\ l_2 = \dfrac{(x-x_1)(x-x_0)}{(x_2-x_1)(x_2-x_0)}$

9. Using quadratic Lagrange Interpolation find the Lagrange interpolating polynomial $P_2(x)$ and hence find value of y at x=2 Given y(0) = 15, y(1) = 48, y(5) = 85

Solution :

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 5$ and $y_0 = 15$, $y_1 = 48$, $y_2 = 85$

$x_1 - x_0 = 1 \neq x_2 - x_1 = 4$

The given data is not equispaced.

$$l_0 = \frac{(x-x_2)(x-x_1)}{(x_0-x_2)(x_0-x_1)} = \frac{(x-5)(x-1)}{(0-5)(0-1)} = \frac{(x^2-6x+5)}{5}$$

$$l_1 = \frac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} = \frac{(x-5)(x-0)}{(1-5)(1-0)} = \frac{(x^2-5x)}{(-4)}$$

$$and\ l_2 = \frac{(x-x_1)(x-x_0)}{(x_2-x_1)(x_2-x_0)} = \frac{(x-1)(x-0)}{(5-1)(5-0)} = \frac{(x^2-x)}{20}$$

$$y = l_0\, y_0 + l_1\, y_1 + l_2\, y_2 = \frac{(x^2-6x+5)}{5}15 + \frac{(x^2-5x)}{(-4)}48 + \frac{(x^2-x)}{20}85$$

$$= -4.75x^2 + 37.75x + 15$$

Which is the Lagrange interpolating polynomial $P_2(x)$

Hence at x=2 the value is $P_2(2) = -4.75(2^2)+37.75(2)+15 = 71.5$

General Lagrange Interpolation is the Interpolation through n given points (x_0,y_0), (x_1,y_1) , (x_2,y_2)...................... , (x_n,y_n) given by the formula

$P_n(x)= l_0\, y_0 + l_1\, y_1 + l_2\, y_2 + \ldots\ldots\ldots\ldots + l_n\, y_n$

Where $l_0 = \dfrac{(x-x_n)\ldots\ldots\ldots\ldots(x-x_2)(x-x_1)}{(x_0-x_n)\ldots\ldots\ldots\ldots(x_0-x_2)(x_0-x_1)}$

$$l_1 = \frac{(x-x_n)\ldots\ldots\ldots\ldots(x-x_2)(x-x_0)}{(x_1-x_n)\ldots\ldots\ldots\ldots(x_1-x_2)(x_1-x_0)}$$

$$l_2 = \frac{(x-x_n)\ldots\ldots\ldots\ldots(x-x_1)(x-x_0)}{(x_2-x_n)\ldots\ldots\ldots\ldots(x_2-x_1)(x_2-x_0)}$$

………

……….

$$and\ l_n = \frac{(x-x_{n-1})\ldots\ldots\ldots\ldots(x-x_1)(x-x_0)}{(x_n-x_{n-1})\ldots\ldots\ldots\ldots(x_2-x_1)(x_2-x_0)}$$

10. Using Lagrange Interpolation find the value of y at x=8

Given y(0) = 18, y(1) = 42, y(7) = 57 and y(9) = 90

Solution :

Here $x_0 = 0, x_1 = 1, x_2 = 7, x_3 = 9$ and $y_0 = 26, y_1 = 40, y_2 = 75, y_3 = 90$

$x_1 - x_0 = 1 \neq x_2 - x_1 = 6$

The given data is not equispaced.

$$l_0 = \frac{(x-x_3)(x-x_2)(x-x_1)}{(x_0-x_3)(x_0-x_2)(x_0-x_1)} = \frac{(8-9)(8-7)(8-1)}{(0-9)(0-7)(0-1)} = \frac{-7}{-63} = \frac{1}{9}$$

$$l_1 = \frac{(x-x_3)(x-x_2)(x-x_0)}{(x_1-x_3)(x_1-x_2)(x_1-x_0)} = \frac{(8-9)(8-7)(8-0)}{(1-9)(1-7)(1-0)} = \frac{(-8)}{(48)} = \frac{1}{6}$$

$$l_2 = \frac{(x-x_3)(x-x_1)(x-x_0)}{(x_2-x_3)(x_2-x_1)(x_2-x_0)} = \frac{(8-9)(8-1)(8-0)}{(7-9)(7-1)(7-0)} = \frac{-56}{-84} = \frac{2}{3}$$

$$and\ l_3 = \frac{(x-x_2)(x-x_1)(x-x_0)}{(x_3-x_2)(x_3-x_1)(x_3-x_0)} = \frac{(8-7)(8-1)(8-0)}{(9-7)(9-1)(9-0)} = \frac{56}{144} = \frac{7}{18}$$

$$y = l_0\, y_0 + l_1\, y_1 + l_2\, y_2 + l_3\, y_3 = \frac{1}{9}(18) + \frac{1}{6}(42) + \frac{2}{3}(57) + \frac{7}{18}(90)$$

$$= 2 + 7 + 38 + 35 = 82$$

Which is the value of y at x=8

Newton divided difference Interpolation (data may not be equispaced)

Newton divided difference Interpolation can be applied to arbitrary spaced data.

The first divided difference is $f[x_0, x_1] = (y_1 - y_0)/(x_1 - x_0)$

$$f[x_1, x_2] = (y_2 - y_1)/(x_2 - x_1)$$

The second divided difference is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

The third divided difference is

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

The nth divided difference is

$$f[x_0, x_1, x_2, x_3, \ldots\ldots, x_n] = \frac{f[x_1, x_2, x_3, \ldots\ldots, x_n] - f[x_0, x_1, x_2, \ldots\ldots, x_{n-1}]}{x_n - x_0}$$

Newton divided difference Interpolation formula is

$$Y = y_0 + (x-x_0)\, f[x_0, x_1] + (x-x_0)(x-x_1)\, f[x_0, x_1, x_2] + \ldots\ldots\ldots\ldots$$

$$+ (x-x_0)(x-x_1) \ldots\ldots\ldots (x-x_{n-1})\, f[x_0, x_1, x_2, \ldots\ldots\ldots\ldots, x_n]$$

Problems

11. Using following data find the Newton's divided difference interpolating polynomial and also find the value of y at x= 15

x	0	6	20	45
y	30	48	88	238

Newton's divided difference table

x	y	First divided difference	Second divided difference	Third divided difference
0	**30**			
		(48-30)/6=**3**		
6	48		(8-3)/11=**0.45**	
		(88-48)/5=8		(0.1 -0.45)/26 = **-0.0136**
11	88		(10-8)/20=0.1	
		(238-88)/15=10		
26	238			

$$Y = y_0 + (x-x_0)\ f[x_0, x_1] + (x-x_0)(x-x_1)\ f[x_0, x_1, x_2]$$

$$+ (x-x_0)(x-x_1)(x-x_2)\ f[x_0, x_1, x_{2.}, x_3]$$

$$= 30 + 3x + x(x-6)(0.45) + x(x-6)(x-11)(-0.0136)$$

The value of y at x= 15

$= 30 + 3(15) + 15(9)(0.45) + 15(9)(4)(-0.0136) = 128.406$

NUMERICAL DIFFERENTIATION

When a function y = f(x) is unknown but its values are given at some points like (x_0, y_0), (x_1, y_1), ... (x_n, y_n) or in form of a table, then we can differentiate using numerical differentiation.

Sometimes it is difficult to differentiate a composite or complicated function which can be done easily in less time and less number of steps by numerical differentiation.

We use following methods for numerical differentiation.

(i) Method based on finite difference operators

(ii) Method based on Interpolation

(i) Method based on finite difference operators

Newton's forward difference Interpolation formula is

$Pn(x) = y_0 + p\Delta y_0 + [p(p-1)/2!]\,\Delta^2 y_0 + [p(p-1)(p-2)/3!]\,\Delta^3 y_0 + \ldots\ldots\ldots\ldots$

where $p = (x - x_0)/h$

Newton's backward difference Interpolation formula is

$Pn(x) = y_n + p\nabla y_n + [p(p+1)/2!]\,\nabla^2 y_n + [p(p+1)(p+2)/3!]\,\nabla^3 y_n$

$+ \ldots\ldots\ldots\ldots + [p(p+1)(p+2)\ldots\ldots(p+n-1)/n!]\,\nabla^n y_n$

where $p = (x - x_n)/h$

Using forward difference the formula for numerical differentiation is

$y'(x_0) = (1/h)\,[\,\Delta y_0 - \Delta^2 y_0/2 + \Delta^3 y_0/3 + \ldots\ldots\ldots\ldots]$

$y''(x_0) = (1/h^2)\,[\,\Delta^2 y_0 - \Delta^3 y_0 + (11/12)\,\Delta^4 y_0 \ldots\ldots\ldots\ldots]$

Using backward difference the formula for numerical differentiation is

$y'(x_n) = (1/h)\,[\,\nabla y_n + \nabla^2 y_n/2 + \nabla^3 y_n/3 + \ldots\ldots\ldots\ldots]$

$y''(x_n) = (1/h^2)\,[\,\nabla^2 y_n + \nabla^3 y_n + (11/12)\,\nabla^4 y_n \ldots\ldots\ldots\ldots]$

If we consider the first term only the formula becomes

$y'(x_0) = (1/h)\,[\,\Delta y_0\,] = (y_1 - y_0)/h$

$y''(x_0) = (1/h^2)\,[\,\Delta^2 y_0\,] = (\Delta y_1 - \Delta y_0)/h^2$

$= [(y_2 - y_1)-(y_1 - y_0)]/h^2 = [y_2 - 2y_1 + y_0]/h^2$

12. **Using following data find the first and second derivative of y at x=0**

x	0	10	20	30	40
y	7	18	32	48	85

Solution

Here $x_0 = 0$, $x_1 = 10$, $x_2 = 20$, $x_3 = 30$, $x_4 = 40$

Forward difference table

x	y	Δ y	Δ^2 y	Δ^3 y	Δ^4 y
0	**7**				
		11			
10	18		**03**		
		14		**02**	
20	32		05		**10**
		19		12	
30	51		17		
		36			
40	87				

Here $x_0 = 0$, $y_0 = 7$, $h = x_1 - x_0 = 10 - 0 = 10$

$\Delta y_0 = 11$, $\Delta^2 y_0 = 3$,

$\Delta^3 y_0 = 2$, $\Delta^4 y_0 = 10$

$p = (x - x_0)/h = (4 - 0)/10 = 0.4$

$y'(x_0) = (1/h) \left[\Delta y_0 - \Delta^2 y_0 /2 + \Delta^3 y_0 /3 - \Delta^4 y_0 /4 + \ldots\ldots\ldots\ldots \right]$

$= 0.1 [11 – 3/2 + 2/3 – 10/4] = 0.7666$

$y''(x_0) = (1/h^2) \left[\Delta^2 y_0 - \Delta^3 y_0 + (11/12) \Delta^4 y_0 \ldots\ldots\ldots \right]$

$= (1/100) [3 - 2 + (11/12) 10] = 0.10166$

(ii) Method based on Interpolation

Linear Interpolation

$$y'(x_0) = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

Quadratic Interpolation

$y'(x_0) = (-3y_0 + 4 y_1 – y_2) /(2h)$

$y'(x_1) = (y_2 - y_0) /(2h)$

$y'(x_2) = (y_0 - 4y_1 + 3y_2)/(2h)$

The second derivative is constant i.e. same at all points because of quadratic interpolation and the interpolating polynomial is of degree two. Hence we must have

$y''(x_0) = (y_0 - 2y_1 + y_2)/(2h)$

$y''(x_1) = (y_0 - 2y_1 + y_2)/(2h)$

$y''(x_2) = (y_0 - 2y_1 + y_2)/(2h)$

Problems

13. **Using following data find the value of first and second derivatives of y at x=30**

x	10	30	50
y	42	64	88

Solution

Here $x_0 = 10, x_1 = 30, x_2 = 50, h = x_1 - x_0 = 30 - 10 = 20$

$y_0 = 42, y_1 = 64, y_2 = 88$

Linear Interpolation

$$y'(x_0) = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{64-42}{30-10} = 1.1$$

Quadratic Interpolation

$y'(x_0) = (-3y_0 + 4y_1 - y_2)/(2h) = [-3(42) + 4(64) - 88]/40 = 1.05$

$y'(x_1) = (y_2 - y_0)/(2h) = (88 - 42)/40 = 1.15$

$y'(x_2) = (y_0 - 4y_1 + 3y_2)/(2h) = (42 - 256 + 264)/40 = 1.25$

$y''(x_0) = (y_0 - 2y_1 + y_2)/(2h) = (42 - 128 + 88)/40 = 0.05$

14. **Using following data find the value of first and second derivatives of y at x=12**

x	0	10	20	30	40
y	7	18	32	48	85

Solution

Here $x_0 = 0, x_1 = 10, \ x_2 = 20, \ x_3 = 30, \ x_4 = 40,$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	**7**				
		11			
10	18		**03**		
		14		**02**	
20	32		05		**10**
		19		12	
30	51		17		
		36			
40	87				

Here $x_0 = 0, \ y_0 = 7, \ h = x_1 - x_0 = 10\text{-}0 = 10$

$\Delta y_0 = 11 , \ \Delta^2 \ y_0 = 3 ,$

$\Delta^3 \ y_0 = 2, \quad \Delta^4 \ y_0 = 10$

$p = (x - x_0)/h \ = (x - 0)/10 \ = 0.1x$

$P_n(x) = y_0 + p\,\Delta y_0 + [\, p(p-1)/2!\,]\,\Delta^2 y_0 + [\, p(p-1)(p-2)/3!\,]\,\Delta^3 y_0$

$+ [\, p(p-1)(p-2)(p\text{-}3)/4!\,]\,\Delta^4 y_0$

$= 7 + 0.1x\,(11) + [0.1x(0.1x - 1)/2!\,]\,(3) + [0.1x(0.1x - 1)(0.1x - 2)/3!\,]\,(2)$

$+ [0.1x(0.1x - 1)(0.1x - 2)(0.1x - 3)/4!\,]\,(10)$

$= 7 + 1.1x + (0.01x^2 - 0.1x)1.5 + (0.001x^3 - 0.03x^2 + 0.2x)/3$

$+ 0.416\,(\ 0.0001x^4 - 0.006x^3 + 0.11x^2 - 0.6x)$

$y = P_n(x) = 0.0000416\,x^4 - 0.0022\,x^3 + 0.05x^2 + 1.26\,x + 7$ ....(1)

Differentiating (1) w.r. to x we get

$y' = 0.0001664\,x^3 - 0.0066\,x^2 + 0.1\,x + 1.26$ .... (2)

$y'(12) = 1.7971392$ at $x = 12$

Differentiating (2) w.r. to x we get

$y'' = 0.0004992 x^2 - 0.0132 x + 0.1$

$y''(12) = 0.0134848$ at $x = 12$

NUMERICAL INTEGRATION

Consider the integral $I = \int_a^b f(x)\, dx$

Where integrand f(x) is a given function and a, b are known which are end points of the interval [a, b]

Either f(x) is given or a table of values of f(x) are given.

Let us divide the interval [a, b] into n number of equal subintervals so that length of each subinterval

is $h = (b - a)/n$

The end points of subintervals are $a = x_0, x_1, x_2, x_3, \ldots, x_n = b$

Trapezoidal Rule of integration

Let us approximate integrand f by a line segment in each subinterval. Then coordinate of end points of subintervals are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Then from x=a to x=b the area under curve of y = f(x) is approximately equal to sum of the areas of n trapezoids of each n subintervals.

So the integral $I = \int_a^b f(x)\, dx = (h/2)[y_0 + y_1] + (h/2)[y_1 + y_2] + (h/2)[y_2 + y_3]$

$+ \ldots + (h/2)[y_{n-1} + y_n]$

$= (h/2)[y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \ldots + y_{n-1} + y_n]$

$= (h/2)[y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})]$

Which is called trapezoidal rule.

The error in trapezoidal rule is $-\frac{b-a}{12} h^2 f''(\theta)$ where $a < \theta < b$

Simpsons rule of Numerical integration (Simpsons 1/3rd rule)

Consider the integral $I = \int_a^b f(x)\, dx$

Where integrand f(x) is a given function and a, b are known which are end points of the interval [a, b]

Either f(x) is given or a table of values of f(x) are given.

Let us approximate integrand f by a line segment in each subinterval. Then coordinate of end points of subintervals are $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots\ldots, (x_n, y_n)$.

We are taking two strips at a time Instead of taking one strip as in trapezoidal rule. For this reason the number of intervals in Simpsons rule of Numerical integration must be even.

The length of each subinterval is h = (b – a)/(2m)

The formula is

$$I = \int_a^b f(x)\,dx = (h/3)\,[\,y_0 + y_{2m} + 4(y_1 + y_3 + \ldots\ldots\ldots + y_{2m-1}) + 2(y_2 + y_4 + \ldots\ldots\ldots + y_{2m-2})\,]$$

The error in Simpson 1/3rd rule is $-\dfrac{b-a}{180} h^4 f^{\prime v}(\theta)$ where $a < \theta < b$

Simpsons rule of Numerical integration (Simpsons 3/8th rule)

Consider the integral $I = \int_a^b f(x)\,dx$

Where integrand f(x) is a given function and a, b are known which are end points of the interval [a, b]

Either f(x) is given or a table of values of f(x) are given.

We are taking three strips at a time Instead of taking one strip as in trapezoidal rule. For this reason the number of intervals in Simpsons $3/8^{th}$ rule of Numerical integration must be multiple of 3.

The length of each subinterval is h = (b – a)/(3m)

The formula is

$$I = \int_a^b f(x)\,dx = (3h/8)\,[\,y_0 + y_{3m} + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{3m-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{3m-3})\,]$$

The error in Simpson 1/3rd rule is $-\dfrac{b-a}{80} h^4 f^{\prime v}(\theta)$ where $a < \theta < b$

15. **Using Trapezoidal and Simpsons rule evaluate the following integral with number of subintervals n =6**

$$\int_0^6 e^{(-x^2)}\,dx$$

Solution:

Here integrand $y = f(x) = \exp(-x^2)$

a=0, b=6, h= (b-a)/n= (6-0)/6=1

x	0	1	2	3	4	5	6
Y= $\exp(-x^2)$	1	e^{-1}	e^{-4}	e^{-9}	e^{-16}	e^{-25}	e^{-36}
	y_0	y_1	y_2	y_3	y_4	Y_5	y_6

(i) **Using Trapezoidal rule**

$$I = (h/2)[\ y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots + y_{n-1})\ ]$$

$$= (1/2)[\ y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\ ]$$

$$= 0.5\ [\ 1 + e^{-36} + 2(e^{-1} + e^{-4} + e^{-9} + e^{-16} + e^{-25})\ ]$$

(ii) **Using Simpsons rule**

$$I = (h/3)\ [\ y_0 + y_{2m} + 4(y_1 + y_3 + \ldots\ldots\ldots + y_{2m-1}) + 2(\ y_2 + y_4 + \ldots\ldots\ldots + y_{2m-2})\ ]$$

$$= (h/3)\ [\ y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(\ y_2 + y_4\ )\ ]$$

$$= (1/3)\ [\ 1 + e^{-36} + 4\ (e^{-1} + e^{-9} + e^{-25}) + 2(e^{-4} + e^{-16}\ )\ ]$$

(iii) **Using Simpsons 3/8th rule**

$$I = (3h/8)\ [\ y_0 + y_{3m} + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{3m-1}) + 2(\ y_3 + y_6 + \ldots\ldots + y_{3m-3})\ ]$$

$$= (3h/8)\ [\ y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(\ y_3\ )\ ]$$

$$= (3/8)\ [\ 1 + e^{-36} + 3\ (e^{-1} + e^{-4} + e^{-16} + e^{-25}) + 2(e^{-9})\ ]$$

16. **Using Trapezoidal and Simpsons rule evaluate the following integral with number of subintervals n =8 and compare the result**

$$\int_0^{0.8} \frac{dx}{4 + x^2}$$

Solution:

Here integrand $y = f(x) = (4 + x^2)^{-1}$

a=0, b= 0.8 , h= (b-a)/n= (0.8-0)/8= 0.1

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Y= $(4 + x^2)^{-1}$	1/4	1/4.01	1/4.04	1/4.09	1/4.16	1/4.25	1/4.36	1/4.49	1/4.64
	y_0	y_1	y_2	y_3	y_4	Y_5	y_6	Y_7	y_8

(i) **Using Trapezoidal rule**

$I = (h/2)[\, y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots\ldots\ldots + y_{n-1})\,]$

$= (0.1/2)[\, y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\,]$

= 0.05 [0.25+ 1/4.64+ 2(1/4.01+ 1/4.04+ 1/4.09+1/4.16+ 1/4.25+ 1/4.36+1/4.49)]

(ii) **Using Simpsons rule**

$I = (h/3)\,[\, y_0 + y_{2m} + 4(y_1 + y_3 + \ldots\ldots\ldots\ldots + y_{2m-1}) + 2(y_2 + y_4 + \ldots\ldots\ldots + y_{2m-2})\,]$

$= (h/3)\,[\, y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\,]$

= (0.1/3) [0.25+ 1/4.64+ 4(1/4.01+ 1/4.09+ 1/4.25+ 1/4.49)

+2(1/4.04+1/4.16+ 1/4.36)]

By direct integration we get

$$\int_0^{0.8} \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^{0.8} = 0.5\left[\tan^{-1}0.4 - \tan^{-1}0\right] = 0.5\tan^{-1}0.4$$

=10.900704743176

Comparing the result we get error in Trapezoidal and Simpsons rule.

17. **Using Trapezoidal and Simpsons rule evaluate the following integral with number of subintervals n =6**

$$I = \int_0^{0.6} \frac{dx}{\sqrt{1+x}}$$

Solution:

Here integrand $y = f(x) = \frac{1}{\sqrt{1+x}}$

a=0, b= 0.6 , h= (b-a)/n = (0.6-0)/6 = 0.1

x	0	0.1	0.2	0.3	0.4	0.5	0.6
Y= $\frac{1}{\sqrt{1+x}}$	1	$\frac{1}{\sqrt{1.1}}$ =0.953462	$\frac{1}{\sqrt{1.2}}$ =0.912871	$\frac{1}{\sqrt{1.3}}$ =0.877058	$\frac{1}{\sqrt{1.4}}$ =0.845154	$\frac{1}{\sqrt{1.5}}$ =0.816496	$\frac{1}{\sqrt{1.6}}$ =0.790569
	y_0	y_1	y_2	y_3	y_4	Y_5	y_6

(i) **Using Trapezoidal rule**

$I = (h/2)[\ y_0 + y_n + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots\ldots + y_{n-1})\ ]$

$= (0.1/2)[\ y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\ ]$

$= 0.05\ [\ 1 + 0.790569 + 2(0.953462 + 0.912871 + 0.877058 + 0.845154 + 0.816496)\ ]$

(ii) **Using Simpsons rule**

$I = (h/3)\ [\ y_0 + y_{2m} + 4(y_1 + y_3 + \ldots\ldots\ldots\ldots + y_{2m-1}) + 2(\ y_2 + y_4 + \ldots\ldots\ldots\ldots + y_{2m-2})\ ]$

$= (h/3)\ [\ y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(\ y_2 + y_4\ )\ ]$

$= (0.1/3)\ [\ 1 + 0.790569 + 4(0.953462 + 0.877058 + 0.816496) + 2(0.912871 + 0.845154)\ ]$

(iii) **Using Simpsons 3/8th rule**

$I = (3h/8)\ [\ y_0 + y_{3m} + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{3m-1}) + 2(\ y_3 + y_6 + \ldots\ldots + y_{3m-3})\ ]$

$= (3h/8)\ [\ y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(\ y_3\ )\ ]$

$= (0.3/8)\ [\ 1 + 0.790569 +$

$3(0.953462 + 0.912871 + 0.845154 + 0.816496) + 2(0.877058\ )]$

UNIT-II

Linear System: Solution by iteration

Gauss-Seidal iteration method

This is an iterative method used to find approximate solution of a system of linear equations.

Some times in iterative method convergence is faster where matrices have large diagonal elements. In this case Gauss elimination method require more number of steps and more row operations. Also sometimes a system has many zero coefficients which require more space to store zeros for example 30 zeros after or before decimal point. In such cases Gauss-Seidal iteration method is very useful to overcome these difficulties and find approximate solution of a system of linear equations.

Procedure:

We shall find a solution x of the system of equations Ax=b with given initial guess x^0.

A is an n x n matrix with non-zero diagonal elements

Step-I Rewrite the given equations in such a way that in first equation coefficient of x_1 is maximum, in second equation coefficient of x_2 is maximum, in third equation coefficient of x_3 is maximum and so on.

Step-II From first equation write x_1 in terms of other variables x_2 , x_3 , x_4 etc.

From the second equation write x_2 in terms of other variables x_1 , x_3 , x_4 etc.

From third equation write x_3 in terms of other variables x_1 , x_2 , x_4 etc.

And so on write all equations in this form.

Step-III

If initial guess is given we take that value otherwise we assume X = (1, 1, 1) as initial guess.

Put $x_2 = 1, x_3 = 1$ in first equation to get x_1 ....(1)

Put x_3 = 1 and put value of x_1 obtained in (1) in the second equation to get value of x_2. ....(2)

Put values of x_1 , x_2 obtained in (1) and (2) in the third equation to get value of x_3.

Step-IV

We repeat this procedure up to desired accuracy and up to desired number of steps.

18. **Solve following linear equations using Gauss-Seidal iteration method starting from 1, 1, 1**

$x_1 + x_2 + 2x_3 = 8$

$2x_1 + 3x_2 + x_3 = 12$

$5x_1 + x_2 + x_3 = 15$

Solution Rewrite the given equations so that each equation for the variable that has coefficient largest we get

$5x_1 + x_2 + x_3 = 15$ ....(1)

$2x_1 + 3x_2 + x_3 = 12$ ....(2)

$x_1 + x_2 + 2x_3 = 10$ ....(3)

From equation (1) we get x_1 in terms of other variables x_2 and x_3 as

$5x_1 = 15 - x_2 - x_3$

$x_1 = (15 - x_2 - x_3)/5 = 3 – 0.2x_2 – 0.2x_3$ ....(4)

From equation (2) we get x_2 in terms of other variables x_1 and x_3 as

$2x_1 + 3x_2 + x_3 = 12$

$x_2 = 4 - (2x_1 + x_3)/3$ ....(5)

From equation (3) we get x_3 in terms of other variables x_1 and x_2 as

$x_1 + x_2 + 2x_3 = 10$

$x_3 = 5 - 0.5x_1 - 0.5x_2$ ....(6)

Step-1

Putting $x_2 = 1$, $x_3 = 1$ in equation (4) we get

$x_1 = 3 – 0.2x_2 – 0.2x_3 = 3 – 0.2 – 0.2 = 2.6$

Putting $x_1 = 2.6$, $x_3 = 1$ in equation (5) we get

$x_2 = 4 - (2x_1 + x_3)/3 = 4 – (5.2+1)/3 = 1.93333$

Putting $x_2 = 1.93333$, $x_1 = 2.6$ in equation (6) we get

$x_3 = 5 - 0.5\,x_1 - 0.5\,x_2 = 5 - 0.5\,(2.6) - 0.5\,(1.93333) = 2.73333$

Step-2

Putting $x_2 = 1.93333$, $x_3 = 2.73333$ in equation (4) we get

$x_1 = 3 - 0.2\,x_2 - 0.2\,x_3 = 3 - 0.2(1.93333) - 0.2\,(2.73333) = 2.066666$

Putting $x_1 = 2.06666$, $x_3 = 2.73333$ in equation (5) we get

$x_2 = 4 - (2x_1 + x_3)/3 = 4 - (4.13333 + 2.73333)/3 = 1.71111$

Putting $x_2 = 1.71111$, $x_1 = 2.066666$ in equation (6) we get

$x_3 = 5 - 0.5\,x_1 - 0.5\,x_2 = 5 - 0.5\,(2.066666) - 0.5\,(1.71111) = 3.11111$

Step-3

Putting $x_2 = 1.71111$, $x_3 = 3.11111$ in equation (4) we get

$x_1 = 3 - 0.2\,x_2 - 0.2\,x_3 = 3 - 0.2(1.71111) - 0.2\,(3.11111) = 2.035555$

Putting $x_1 = 2.035555$, $x_3 = 3.11111$in equation (5) we get

$x_2 = 4 - (2x_1 + x_3)/3 = 4 - (4.07111 + 3.11111)/3 = 1.605925$

Putting $x_2 = 1.605925$, $x_1 = 2.035555$ in equation (6) we get

$x_3 = 5 - 0.5\,x_1 - 0.5\,x_2 = 5 - 0.5\,(2.035555) - 0.5\,(1.605925) = 3.17926$

Step-4

Putting $x_2 = 1.605925$, $x_3 = 3.17926$ in equation (4) we get

$x_1 = 3 - 0.2\,x_2 - 0.2\,x_3 = 3 - 0.2(1.605925) - 0.2\,(3.17926) = 2.042962$

Putting $x_1 = 2.042962$, $x_3 = 3.17926$ in equation (5) we get

$x_2 = 4 - (2x_1 + x_3)/3 = 4 - (4.08592 + 3.17926)/3 = 1.57827$

Putting $x_2 = 1.57827$, $x_1 = 2.042962$ in equation (6) we get

$x_3 = 5 - 0.5\,x_1 - 0.5\,x_2 = 5 - 0.5\,(2.042962) - 0.5\,(1.57827) = 3.18938$

Eigen values and Eigen vectors by Power method

This is an iterative method used to find approximate value of Eigen values and Eigen vectors of an n x n non-singular matrix A.

Procedure:

We start with any non-zero vector x_0 of n components and compute followings.

$x_1 = Ax_0$

$x_2 = A x_1$

$x_3 = A x_2$

....

....

....

$x_n = A x_{n-1}$

For any n x n non-singular matrix A we can apply this method and we get a dominant

eigen value λ such that absolute value of this eigen value λ is greater than that of other eigen values.

Theorem: Let A be an n x n real symmetric matrix. Let $x \neq 0$ be any real vector with n components. Let $y=Ax$, $m_0 = x^T x$, $m_1=x^Ty$, $m_2=y^Ty$

Then the ratio $r = m_1 / m_0$ called Rayleigh quotient is an approximate eigen value λ of A.

Assuming $r = \lambda - \epsilon$ we have $|\epsilon| \leq \sqrt{\frac{m_2}{m_0} - r^2}$

where ϵ is the error of ratio $r = m_1 / m_0$

19 . **Find the eigen values and eigen vectors of the matrix** $\begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}$ **by Power method taking** $x_0 = [\,1\ 1\,]^T$

Solution Let $A = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}$. Given $x_0 = [\,1\ \ 1\,]^T$

$x_1 = Ax_0$

$= \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 5/9 \end{bmatrix}$

Dominated eigen value is 9 and and eigen vector is $\begin{bmatrix} 1 \\ 5/9 \end{bmatrix}$

$x_2 = A x_1$

$= \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 5/9 \end{bmatrix} = \begin{bmatrix} 7.666 \\ 4.111 \end{bmatrix} = 7.666 \begin{bmatrix} 1 \\ 0.536 \end{bmatrix}$

Dominated eigen value is 7.666 and and eigen vector is $\begin{bmatrix} 1 \\ 0.536 \end{bmatrix}$

$x_3 = A x_2$

$= \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.536 \end{bmatrix} = \begin{bmatrix} 7.608 \\ 4.072 \end{bmatrix} = 7.608 \begin{bmatrix} 1 \\ 0.535 \end{bmatrix}$

Dominated eigen value is 7.608 and and eigen vector is $\begin{bmatrix} 1 \\ 0.535 \end{bmatrix}$

20 . **Find the eigen values and eigen vectors of the matrix** $\begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}$ **by Power method taking $x_0 = [\,1\ \ 1\ \ 1\,]^T$**

Solution Let $A = \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}$. Given $x_0 = [\,1\ \ 1\ \ 1]^T$

$x_1 = Ax_0$

$= \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 10 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}$

Dominated eigen value is 10 and and eigen vector is $\begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}$

$x_2 = A x_1$

$= \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.5 \\ 4 \\ 8 \end{bmatrix} = 8.5 \begin{bmatrix} 1 \\ 0.4705 \\ 0.9411 \end{bmatrix}$

Dominated eigen value is 8.5 and and eigen vector is $\begin{bmatrix} 1 \\ 0.4705 \\ 0.9411 \end{bmatrix}$

$x_3 = A\, x_2$

$$= \begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4705 \\ 0.9411 \end{bmatrix} = \begin{bmatrix} 8.3526 \\ 3.941 \\ 7.5875 \end{bmatrix} = 8.3526 \begin{bmatrix} 1 \\ 0.4718 \\ 0.9084 \end{bmatrix}$$

Dominated eigen value is 8.3526 and and eigen vector is $\begin{bmatrix} 1 \\ 0.4718 \\ 0.9084 \end{bmatrix}$

Unit III: Solution of IVP by Euler's method, Heun's method and Runge-Kutta fourth order method. Basic concept of optimization, Linear programming, simplex method, degeneracy, and Big-M method.

Numerical Solution of Differential Equation:

Introduction:

We consider the first order differential equation

$$y' = f(x, y)$$

With the initial condition

$$y\,(x_o) = y_0$$

The sufficient conditions for the existence of unique solution on the interval $[x_0, b]$ are the well-known Lipschitz conditions. However in 'Numerical Analysis', one finds values of y at successive steps, $x = x_1, x_2, \dots, x_n$ with spacing h. There are many numerical methods available to find solution of IVP, such as : Picards method, Euler's method, Taylor' series method, Runge-Kutta method etc.

In the present section we will solve the ode

$y' = f(x, y), \quad y(x_0) = y_0$ in the interval $I = (x_0, x_n)$ (1)

using a numerical scheme applied to discrete node $x_n = x_0 + nh$, where h is the step-size by Euler's method, Heun's method and Runge-Kutta method.

- In Euler's method we use the slope evaluated at the current level (x_n, y_n) and use that value as an approximation of the slope throughout the interval (x_n, x_{n+1}).
- Hune' method samples the slope at beginning and at the end and uses the average as the final approximation of the slope. It is also known as Runge-kutta method of order-2.
- Runge-kutta method of order-4 improve on Euler' s method looking at the slope at multiple points.

The necessary formula for solution of (1) by **Euler' s method** is:

$y_{j+1} = y_j + hf(x_j, y_j)$, j = 0, 1, 2, … n - 1.

The necessary formula for solution of (1) by **Hune' s method** is:

$y_{j+1} = y_j + \frac{1}{2}(k_1 + k_2)$, j = 0, 1, 2, … n - 1.

Where $k_1 = hf(x_j, y_j), k_2 = hf(x_j + h, y_j + k_1)$

The necessary formula for solution of (1) by **Runge – Kutta method of order-4** is:

$y_{j+1} = y_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, j = 0, 1, 2, …, n – 1.

Where $k_1 = hf(x_j, y_j)$

$$k_2 = hf(x_j + \frac{1}{2}h, y_j + \frac{1}{2}k_1)$$

$$k_3 = hf(x_j + \frac{1}{2}h, y_j + \frac{1}{2}k_2)$$

$$k_4 = hf(x_j + h, y_j + k_3)$$

Example : Use the Euler method to solve numerically the initial value problem

$$u' = -2tu^2, u(0) = 1$$

With h = 0.2 on the interval [0, 1]. Compute u (1.0)

We have

$u_{j+1} = u_j - 2ht_j u_j^2, \quad j = 0,1,2,3,4.$ *[Here x and y are replaced by t and u respectively]*

With h=0.2. The initial condition gives $u_0=1$

For j = 0: $t_0 = 0$, $u_0 = 1$

u (0.2) = $u_1 = u_0 - 2ht_0u_0^2 = 1.0$.

For j = 1: $t_1 = 0.2$, $u_1 = 1$

u (0.4) = $u_2 = u_1 - 2ht_1u_1^2 = 0.92$.

For j = 2: $t_2 = 0.4$, $u_2 = 0.92$

u (0.6) = $u_3 = u_2 - 2ht_2u_2^2 = 0.78458$.

For j = 3: $t_3 = 0.6$, $u_3 = 0.78458$

u(0.8) = $u_4 = 0.63684$.

Similarly, we get

u(1.0) = $u_5 = 0.50706$.

Note: In the similar way IVP can be solved by Heun's method and Runge-Kutta fourth order method.

Optimization

Optimization is the means by which scarce resources can be utilized in an efficient manner so as to maximize the profit or minimize the loss.

Basic components of an optimization problem:

An **objective function** expresses the main aim of the model which is either to be minimized or maximized. For example, in a manufacturing process, the aim may be to *maximize the profit* or *minimize the cost.* In comparing the data prescribed by a user-defined model with the observed data, the aim is *minimizing the total deviation* of the predictions based on the model from the observed data. In designing a bridge, the goal is to *maximize the strength* and *minimize size.*

A set of **unknowns** or **variables** control the value of the objective function. In the manufacturing problem, the variables may include the *amounts of different resources used* or the *time spent on each activity.* In fitting-the-data problem, the unknowns are the *parameters* of the model. In the pier design problem, the variables are the *shape and dimensions* of the pier.

A set of **constraints** are those which allow the unknowns to take on certain values but exclude others. In the manufacturing problem, one cannot spend negative amount of time on any activity, so one constraint is that the "time" variables are to be non-negative. In the pier design problem, one would probably want to limit the breadth of the base and to constrain its size.

The optimization problem is then to find values of the variables that minimize or maximize the objective function while satisfying the constraints.

Objective Function

As already stated, the objective function is the mathematical function one wants to maximize or minimize, subject to certain constraints. Many optimization problems have a single

In the present context we will apply the optimization technique to Linear programming problem.

The general form of a linear programming problem is:

Maximize(Minimize) $z = c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to the constraints:

$$c_{11}x_1 + c_{12}\,x_2 + \ldots + c_{1n}x_n \leq / \geq / = b_1$$

$$c_{21}x_1 + c_{22}\,x_2 + \ldots + c_{2n}x_n \leq / \geq / = b_2$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$c_{m1}x_1 + c_{m2}\,x_2 + \ldots + c_{mn}x_n \leq / \geq / = b_m$$

$$x_1, x_2, \ldots x_n \geq 0$$

In short, Maximize(Minimize) $Z = CX$ .... (1)

Subject to the constraints: $AX \leq / \geq / = B$ .... (2)

$X \geq 0$ .... (3)

Where the expression under (1), (2) and (3) are known as objective function, constraints and non-negativity restrictions respectively.

The **problem definition and formulation** includes the steps: identification of the decision variables; formulation of the model objective(s) and the formulation of the model constraints. In performing these steps the following are to be considered.

1. Identify the important elements that the problem.
2. Determine the number of independent variables, the number of equations required to describe the system, and the number of unknown parameters.

Graphical Method

To solve *Linear Programming problem* (LPP), *Graphical method* helps to visualize the procedure explicitly. It also helps to understand the different terminologies associated with the solution of LPP. Let us discuss these aspects with the help of an example. However, this visualization is possible for a maximum of two decision variables. Thus, a LPP with two decision variables is opted for discussion. However, the basic principle remains the same for more than two decision variables also, even though the visualization beyond two-dimensional case is not easily possible.

Let us consider the same LPP (general form) discussed in previous class, stated here once again for convenience.

Maximize	$Z = 6x + 5y$	
subject to	$2x - 3y \le 5$	(C −1)
	$x + 3y \le 11$	(C − 2)
	$4x + y \le 15$	(C −3)
	$x, y \ge 0$	(C − 4) & (C −5)

First step to solve above LPP by graphical method, is to plot the inequality constraints one-by-one on a graph paper. Fig. 1a shows one such plotted constraint.

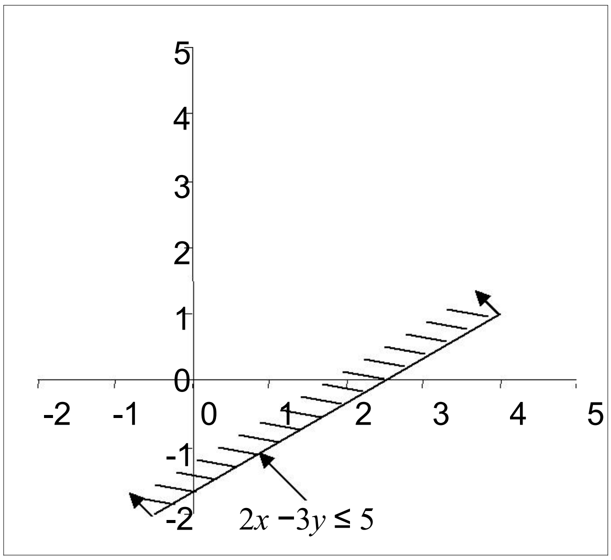

Fig. 1a Plot showing first constraint ($2x - 3y \le 5$)

Fig. 1b shows all the constraints including the nonnegativity of the decision variables (i.e., $x \ge 0$ and $y \ge 0$).

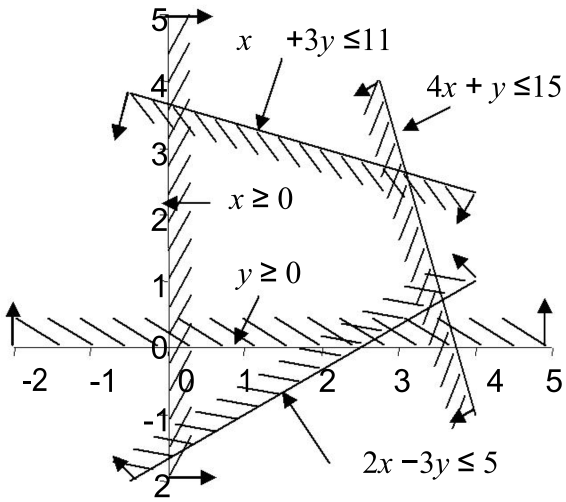

Fig. 1b Plot of all the constraints

Common region of all these constraints is known as *feasible region* (Fig. 1c). Feasible region implies that each and every point in this region satisfies all the constraints involved in the LPP.

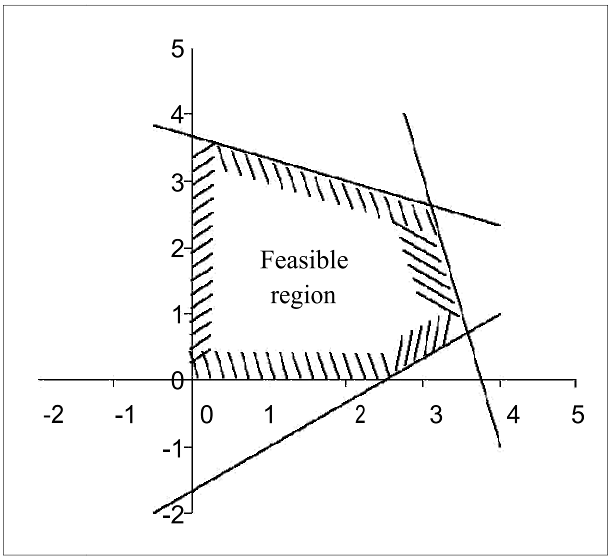

Fig. 1c Feasible region

Once the feasible region is identified, objective function ($Z = 6x + 5y$) is to be plotted on it. As the (optimum) value of Z is not known, objective function is plotted by considering any constant, k (Fig. 1d). The straight line, $6x + 5y = k$ (constant), is known as *Z line* (Fig. 1d). This line can be shifted in its perpendicular direction (as shown in the Fig. 1d) by changing the value of k. Note that, position of *Z line* shown in Fig. 1d, showing the intercept, c, on the y axis is 3. If, $6x + 5y = k \Rightarrow 5y = -6x + k \Rightarrow y = \frac{-6}{5}x + \frac{k}{5}$, i.e., $m = \frac{-6}{5}$ and $c = \frac{k}{5} = 3 \Rightarrow k = 15$.

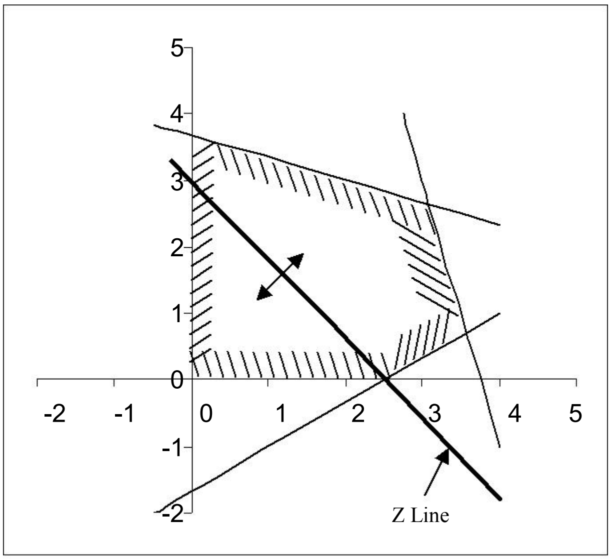

Fig. 1d Plot of *Z line* and *feasible region*

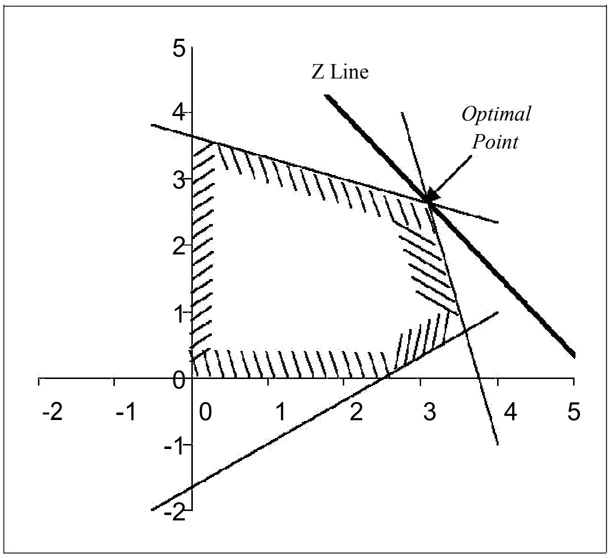

Fig. 1e Location of Optimal Point

Now it can be visually noticed that value of the objective function will be maximum when it passes through the intersection of $x + 3y = 11$ and $4x + y = 15$ (straight lines associated with the second and third inequality constraints). This is known as *optimal point* (Fig. 1e). Thus the *optimal point* of the present problem is $x^* = 3.091$ and $y^* = 2.636$. And the optimal solution is $= 6x^* + 5y^* = 31.727$

Visual representation of different cases of solution of LPP

A linear programming problem may have i) a unique, finite solution, ii) an unbounded solution iii) multiple (or infinite) number of optimal solutions, iv) infeasible solution and v) a unique feasible point. In the context of graphical method it is easy to visually demonstrate the different situations which may result in different types of solutions.

Unique, finite solution

The example demonstrated above is an example of LPP having a unique, finite solution. In such cases, optimum value occurs at an extreme point or vertex of the feasible region.

Unbounded solution

If the feasible region is not bounded, it is possible that the value of the objective function goes on increasing without leaving the feasible region. This is known as unbounded solution (Fig 2).

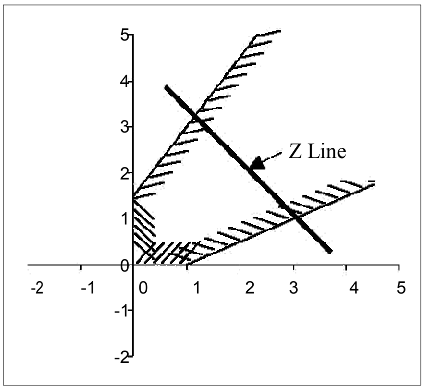

Fig. 2 Unbounded Solution

Multiple (infinite) solutions

If the *Z line* is parallel to any side of the feasible region all the points lying on that side constitute optimal solutions as shown in Fig 3.

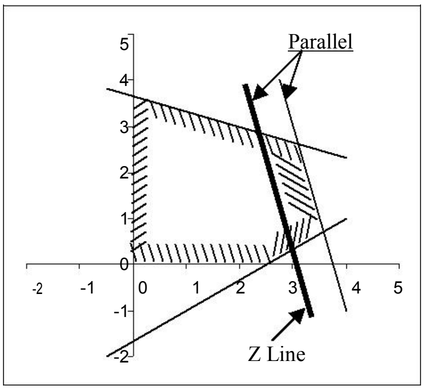

Fig. 3 Multiple (infinite) Solution

Infeasible solution

Sometimes, the set of constraints does not form a feasible region at all due to inconsistency in the constraints. In such situation the LPP is said to have infeasible solution. Fig 4 illustrates such a situation.

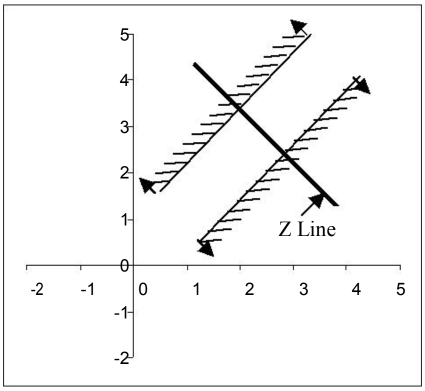

Fig. 4 Infeasible Solution

Unique feasible point

This situation arises when feasible region consist of a single point. This situation may occur only when number of constraints is at least equal to the number of decision variables. An example is shown in Fig 5. In this case, there is no need for optimization as there is only one solution.

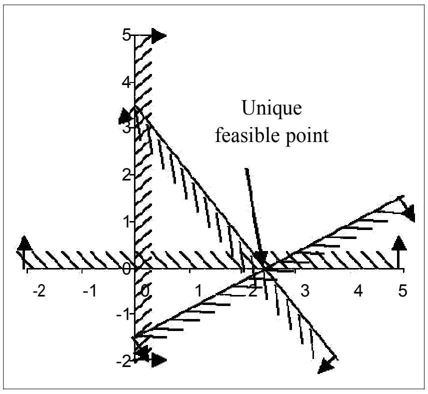

Fig. 5 Unique feasible point

Simplex method

Recall from the previous discussion that the optimal solution of a LPP, if exists, lies at one of the vertices of the feasible region. Thus one way to find the optimal solution is to find all the basic feasible solutions of the standard form and investigate them one-by-one to get at the optimal. However, again recall thatt, for m equations with n variables there exists a huge number (${}^{n}c_{m}$) of basic feasible solutions. In such a case, inspection of all the solutions one-by-one is not practically feasible. However, this can be overcome by simplex method. Conceptual principle of this method can be easily understood for a three dimensional case (however, simplex method is applicable for any higher dimensional case as well).

Imagine a feasible region (i.e., volume) bounded by several surfaces. Each vertex of this volume, which is a basic feasible solution, is connected to three other adjacent vertices by a straight line to each being the intersection of two surfaces. Being at any one vertex (one of the basic feasible solutions), *simplex algorithm* helps to move to another adjacent vertex which is closest to the optimal solution among all the adjacent vertices. Thus, it follows the shortest route to reach the optimal solution from the starting point. It can be noted that the shortest route consists of a sequence of basic feasible solutions which is generated by *simplex algorithm*.

Simplex algorithm

Simplex algorithm is discussed using an example of LPP. Let us consider the following problem.

$$\begin{aligned} \text{Maximize} \quad & Z = 4x_1 - x_2 + 2x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + 2x_3 \le 6 \\ & x_1 - 4x_2 + 2x_3 \le 0 \\ & 5x_1 - 2x_2 - 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Simplex algorithm is used to obtain the solution of this problem. First let us transform the LPP to its standard form as shown below.

$$\begin{aligned} \text{Maximize} \quad & Z = 4x_1 - x_2 + 2x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + 2x_3 + x_4 = 6 \\ & x_1 - 4x_2 + 2x_3 + x_5 = 0 \\ & 5x_1 - 2x_2 - 2x_3 + x_6 = 4 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

It can be recalled that x_4 , x_5 and x_6 are slack variables. Above set of equations, including the objective function can be transformed to canonical form as follows:

$$\begin{aligned} -4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6 + Z &= 0 \\ 2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 &= 6 \\ x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 &= 0 \\ 5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 &= 4 \end{aligned}$$

The basic solution of above canonical form is $x_4 = 6$, $x_5 = 0$, $x_6 = 4$, $x_1 = x_2 = x_3 = 0$ and $Z = 0$. It can be noted that, x_4 , x_5 and x_6 are known as basic variables and x_1 , x_2 and x_3 are known as nonbasic variables of the canonical form shown above. Let us denote each equation of above canonical form as:

$$\begin{aligned} (Z) \quad & -4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6 + Z = 0 \\ (x_4) \quad & 2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 = 6 \\ (x_5) \quad & x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 = 0 \\ (x_6) \quad & 5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 = 4 \end{aligned}$$

For the ease of discussion, right hand side constants and the coefficients of the variables are symbolized as follows:

The left-most column is known as *basis* as this is consisting of basic variables. The coefficients in the first row ($c_1 \wedge c_6$) are known as *cost coefficients*. Other subscript notations are self explanatory and used for the ease of discussion. For each coefficient, first subscript indicates the subscript of the basic variable in that equation. Second subscript indicates the subscript of variable with which the coefficient is associated. For example, c_{52} is the coefficient of x_2 in the equation having the basic variable x_5 with nonzero coefficient (i.e., c_{55} is nonzero).

This completes first step of calculation. After completing each step (iteration) of calculation, three points are to be examined:

1. Is there any possibility of further improvement?
2. Which nonbasic variable is to be entered into the basis?
3. Which basic variable is to be exited from the basis?

The procedure to check these points is discussed next.

4. **Is there any possibility of further improvement?**

   If any of the cost coefficients is negative, further improvement is possible. In other words, if all the cost coefficients are nonnegative, the basic feasible solution obtained in that step is optimum.

5. **Which nonbasic variable is to be entered?**

   Entering nonbasic variable is decided such that the unit change of this variable should have maximum effect on the objective function. Thus the variable having the coefficient which is minimum among all the cost coefficients is to be entered, i.e., x_S is to be entered if cost coefficient c_S is minimum.

6. **Which basic variable is to be exited?**

   After deciding the entering variable x_S, x_r (from the set of basic variables) is decided to be the exiting variable if $\frac{b_r}{c_{rs}}$ is minimum for all possible *r*, provided c_{rs} is positive.

   It can be noted that, c_{rs} is considered as pivotal element to obtain the next canonical form.

In this example, $c_1\ (=-4)$ is the minimum. Thus, x_1 is the entering variable for the next step of calculation. *r* may take any value from 4, 5 and 6. It is found that $\frac{b_4}{c_{41}} = \frac{6}{2} = 3$, $\frac{b_5}{c_{51}} = \frac{0}{1} = 0$ and $\frac{b_6}{c_{61}} = \frac{4}{5} = 0.8$. As, $\frac{b_5}{c_{51}}$ is minimum, *r* is 5. Thus x_5 is to be exited and c_{51} is the pivotal element and x_5 is replaced by x_1 in the basis. Set of equations are transformed through pivotal operation to another canonical form considering c_{51} as the pivotal element. The procedure of pivotal operation is already explained in first class. However, as a refresher it is explained here once again.

1. Pivotal row is transformed by dividing it with the pivotal element. In this case, pivotal element is 1.
2. For other rows: Let the coefficient of the element in the pivotal column of a particular row be "*l*". Let the pivotal element be "*m*". Then the pivotal row is multiplied by *l* / *m* and then subtracted from that row to be transformed. This operation ensures that the coefficients of the element in the pivotal column of that row becomes zero, e.g., *Z row*: l = -4 , m = 1. So, pivotal row is multiplied by l / m = -4 / 1 = -4, obtaining

   $$-4x_1 + 16x_2 - 8x_3 + 0x_4 - 4x_5 + 0x_6 = 0$$

   This is subtracted from *Z row* obtaining,

   $$0x_1 - 15x_2 + 6x_3 + 0x_4 + 4x_5 + 0x_6 + Z = 0$$

   The other two rows are also suitably transformed.

After the pivotal operation, the canonical form obtained is shown below.

$$\begin{array}{llr}
(Z) & 0x_1 - 15x_2 + 6x_3 + 0x_4 + 4x_5 + 0x_6 + Z & = 0 \\
(x_4) & 0x_1 + 9x_2 - 2x_3 + 1x_4 - 2x_5 + 0x_6 & = 6 \\
(x_1) & 1x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 & = 0 \\
(x_6) & 0x_1 + 18x_2 - 12x_3 - 0x_4 - 5x_5 + 1x_6 & = 4
\end{array}$$

The basic solution of above canonical form is $x_1 = 0$, $x_4 = 6$, $x_6 = 4$, $x_3 = x_4 = x_5 = 0$ and $Z = 0$. However, this is not the optimum solution as the cost coefficient c_2 is negative. It is observed that c_2 (= -15) is minimum. Thus, $s = 2$ and x_2 is the entering variable. *r* may take any value from 4, 1 and 6. However, c_{12} $(= -4)$ is negative. Thus, *r* may be either 4 or 6. It is found that, $\frac{b_4}{c_{42}} = \frac{6}{9} = 0.667$, and $\frac{b_6}{c_{62}} = \frac{4}{18} = 0.222$. As $\frac{b_6}{c_{62}}$ is minimum, *r* is 6 and x_6 is to be exited from the basis. c_{62} (=18) is to be treated as pivotal element. The canonical form for next iteration is as follows:

$$\begin{array}{llr}
(Z) & 0x_1 + 0x_2 - 4x_3 + 0x_4 - \frac{1}{6}x_5 + \frac{5}{6}x_6 + Z & = \frac{10}{3} \\
(x_4) & 0x_1 + 0x_2 + 4x_3 + 1x_4 + \frac{1}{2}x_5 - \frac{1}{2}x_6 & = 4 \\
(x_1) & 1x_1 + 0x_2 - \frac{2}{3}x_3 + 0x_4 - \frac{1}{9}x_5 + \frac{2}{9}x_6 & = \frac{8}{9} \\
(x_2) & 0x_1 + 1x_2 - \frac{2}{3}x_3 + 0x_4 - \frac{5}{18}x_5 + \frac{1}{18}x_6 & = \frac{2}{9}
\end{array}$$

The basic solution of above canonical form is $x_1 = \frac{8}{9}$, $x_2 = \frac{2}{9}$, $x_4 = 4$, $x_2 = x_3 = x_5 = 0$ and $Z = \frac{10}{3}$.

It is observed that c_3 (= - 4) is negative. Thus, optimum is not yet achieved. Following similar procedure as above, it is decided that x_3 should be entered in the basis and x_4 should be exited from the basis. Thus, x_4 is replaced by x_3 in the basis. Set of equations are transformed to another canonical form considering c_{43} (= 4) as pivotal element. By doing so, the canonical form is shown below.

$$
\begin{array}{ll}
(Z) & 0x_1 + 0x_2 + 0x_3 + 1x_4 + \frac{1}{3}x_5 + \frac{1}{3}x_6 + Z = \frac{22}{3} \\
(x_3) & 0x_1 + 0x_2 + 1x_3 + \frac{1}{4}x_4 + \frac{1}{8}x_5 - \frac{1}{8}x_6 = 1 \\
(x_1) & 1x_1 + 0x_2 + 0x_3 + \frac{1}{6}x_4 - \frac{1}{36}x_5 + \frac{5}{36}x_6 = \frac{14}{9} \\
(x_2) & 0x_1 + 1x_2 + 0x_3 + \frac{1}{6}x_4 - \frac{7}{36}x_5 - \frac{1}{36}x_6 = \frac{8}{9}
\end{array}
$$

The basic solution of above canonical form is $x_1 = \frac{14}{9}$, $x_2 = \frac{8}{9}$, $x_3 = 1$, $x_4 = x_5 = x_6 = 0$ and $Z = \frac{22}{3}$.

It is observed that all the cost coefficients are positive. Thus, optimum is achieved. Hence, the optimum solution is

$$Z = \frac{22}{3} = 7.333$$

$$x_1 = \frac{14}{9} = 1.556$$

$$x_2 = \frac{8}{9} = 0.889$$

$$x_3 = 1$$

The calculation shown above can be presented in a tabular form, which is known as *Simplex Tableau*. Construction of *Simplex Tableau* will be discussed next.

Construction of Simplex Tableau

Same LPP is considered for the construction of *simplex tableau*. This helps to compare the calculation shown above and the construction of *simplex tableau* for it.

After preparing the canonical form of the given LPP, simplex tableau is constructed as follows.

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	x_5	x_6		
1	Z	1	-4	1	-2	0	0	0	0	--
	x_4	0	2	1	2	1	0	0	6	3
	x_5	0	1	-4	2	0	1	0	0	0
	x_6	0	5	-2	-2	0	0	1	4	$\frac{4}{5}$

Pivotal Row Pivotal Column

Pivotal Element

After completing each iteration, the steps given below are to be followed. Logically, these steps are exactly similar to the procedure described earlier. However, steps described here are somewhat mechanical and easy to remember!

Check for optimum solution:

1. Investigate whether all the elements in the first row (i.e., Z row) are nonnegative or not. Basically these elements are the coefficients of the variables headed by that column. If all such coefficients are nonnegative, optimum solution is obtained and no need of further iterations. If any element in this row is negative, the operation to obtain simplex tableau for the next iteration is as follows:

Operations to obtain next simplex tableau:

2. The entering variable is identified (described earlier). The corresponding column is marked as *Pivotal Column* as shown above.
3. The exiting variable from the basis is identified (described earlier). The corresponding row is marked as *Pivotal Row* as shown above.
4. Coefficient at the intersection of *Pivotal Row* and *Pivotal Column* is marked as *Pivotal Element* as shown above.
5. In the basis, the exiting variable is replaced by entering variable.
6. All the elements in the pivotal row are divided by pivotal element.
7. For any other row, an elementary operation is identified such that the coefficient in the pivotal column in that row becomes zero. The same operation is applied for all other elements in that row and the coefficients are changed accordingly. A similar procedure is followed for all other rows.

   For example, say, (2 x *pivotal element* + *pivotal coefficient in first row*) produce zero in the pivotal column in first row. The same operation is applied for all other elements in the first row and the coefficients are changed accordingly.

Simplex tableaus for successive iterations are shown below. *Pivotal Row*, *Pivotal Column* and *Pivotal Element* for each tableau are marked as earlier for the ease of understanding.

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	x_5	x_6		
2	Z	1	0	-15	6	0	4	0	0	--
	x_4	0	0	9	-2	1	-2	0	6	1/3
	x_1	0	1	-4	2	0	1	0	0	--
	x_6	0	0	18	-12	0	-5	1	4	2/9

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Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	x_5	x_6		
3	Z	1	0	0	-4	0	$-\frac{1}{6}$	$\frac{5}{6}$	$\frac{10}{3}$	--
	x_4	0	0	0	4	1	$\frac{1}{2}$	$-\frac{1}{2}$	4	1
	x_1	0	1	0	$-\frac{2}{3}$	0	$-\frac{1}{9}$	$\frac{2}{9}$	$\frac{8}{9}$	--
	x_2	0	0	1	$-\frac{2}{3}$	0	$-\frac{5}{18}$	$\frac{1}{18}$	$\frac{2}{9}$	--
4	Z	1	0	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{22}{3}$	
	x_3	0	0	0	1	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{8}$	1	
	x_1	0	1	0	0	$\frac{1}{6}$	$-\frac{1}{36}$	$\frac{2}{9}$	$\frac{14}{9}$	
	x_2	0	0	1	0	$\frac{1}{6}$	$-\frac{7}{36}$	$-\frac{1}{36}$	$\frac{8}{9}$	

Optimum value of Z

Value of x_3

All the coefficients are nonnegative. Thus optimum solution is achieved.

Value of x_1

Value of x_2

As all the elements in the first row (i.e., Z row), at iteration 4, are nonnegative, optimum solution is achieved. Optimum value of Z is 22/3 as shown above. Corresponding values of basic variables are $x_1 = \frac{14}{9} = 1.556$, $x_2 = \frac{8}{9} = 0.889$, $x_3 = 1$ and those of nonbasic variables are all zero (i.e., $x_4 = x_5 = x_6 = 0$).

It can be noted that at any iteration the following two points must be satisfied:

1. All the basic variables (other than Z) have a coefficient of zero in the Z row.

2. Coefficients of basic variables in other rows constitute a unit matrix.

If any of these points are violated at any iteration, it indicates a wrong calculation. However, reverse is not true.

Big-M method

Introduction

In the previous lecture the *simplex method* was discussed with required transformation of objective function and constraints. However, all the constraints were of inequality type with 'less-than-equal-to' ($\le$) sign. However, 'greater-than-equal-to' ($\ge$) and 'equality' ($=$) constraints are also possible. In such cases, a modified approach is followed, which will be discussed in this lecture. Different types of LPP solutions in the context of Simplex method will also be discussed. Finally, a discussion on minimization vs maximization will be presented.

Simplex Method with 'greater-than-equal-to' ($\ge$) and equality ($=$) constraints

The LP problem, with 'greater-than-equal-to' ($\ge$) and equality ($=$) constraints, is transformed to its standard form in the following way.

- One 'artificial variable' is added to each of the 'greater-than-equal-to' ($\ge$) and equality ($=$) constraints to ensure an initial basic feasible solution.
- Artificial variables are 'penalized' in the objective function by introducing a large negative (positive) coefficient M for maximization (minimization) problem.
- Cost coefficients, which are supposed to be placed in the Z-row in the initial simplex tableau, are transformed by 'pivotal operation' considering the column of artificial variable as 'pivotal column' and the row of the artificial variable as 'pivotal row'.
- If there are more than one artificial variable, step 3 is repeated for all the artificial variables one by one.

Let us consider the following LP problem

$$
\begin{array}{ll}
\text{Maximize} & Z = 3x_1 + 5x_2 \\
\text{subject to} & x_1 + x_2 \ge 2 \\
 & x_2 \le 6 \\
 & 3x_1 + 2x_2 = 18 \\
 & x_1\,,\, x_2 \ge 0
\end{array}
$$

After incorporating the artificial variables, the above LP problem becomes as follows:

$$
\begin{aligned}
&\text{Maximize} \quad Z = 3x_1 + 5x_2 - Ma_1 - Ma_2 \\
&\text{subject to } x_1 + x_2 - x_3 + a_1 = 2 \; x_2 + \\
&\qquad x_4 = 6 \\
&\qquad 3x_1 + 2x_2 + a_2 = 18 \\
&\qquad x_1 \; , x_2 \geq 0
\end{aligned}
$$

where x_3 is surplus variable, x_4 is slack variable and a_1 and a_2 are the artificial variables. Cost coefficients in the objective function are modified considering the first constraint as follows:

$$
\begin{aligned}
&Z - 3x_1 - 5x_2 + Ma_1 + Ma_2 = 0 \qquad (E_1) \\
&x_1 + x_2 - x_3 + a_1 \qquad = 2 \qquad (E_2) \leftarrow \text{Pivotal Row}
\end{aligned}
$$

Pivotal Column

Thus, pivotal operation is $E_1 - M \cdot E_2$, which modifies the cost coefficients as follows:

$$Z - (3 + M)x_1 - (5 + M)x_2 + Mx_3 + 0\,a_1 + Ma_2 = -2M$$

Next, the revised objective function is considered with third constraint as follows:

$$
\begin{aligned}
&Z - (3 + M)x_1 - (5 + M)\,x_2 + Mx_3 + 0\,a_1 + Ma_2 = -2M \qquad (E_3) \\
&3\,x_1 \qquad + 2\,x_2 \qquad\qquad + a_2 = 18 \qquad (E_4) \leftarrow \text{Pivotal Row}
\end{aligned}
$$

Pivotal Column

Obviously pivotal operation is $E_3 - M \cdot E_4$, which further modifies the cost coefficients as follows:

$$Z - (3 + 4M)x_1 - (5 + 3M)x_2 + Mx_3 + 0\,a_1 + 0\,a_2 = -20M$$

The modified cost coefficients are to be used in the Z-row of the first simplex tableau.

Next, let us move to the construction of simplex tableau. Pivotal column, pivotal row and pivotal element are marked (same as used in the last class) for the ease of understanding.

Iteration	Basis	Z	x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
			Variables							
1	Z	1	$-3-4M$	$-5-3M$	M	0	0	0	$-20M$	--
	a_1	0	1	1	-1	0	1	0	2	2
	x_4	0	0	1	0	1	0	0	6	--
	a_2	0	3	2	0	0	0	1	18	6

Note that while comparing $(-3-4M)$ and $(-5-3M)$, it is decided that $(-3-4M) < (-5-3M)$ as M is any arbitrarily large number.

Successive iterations are shown as follows:

Iteration	Basis	Z	x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
			Variables							
2	Z	1	0	$-2+M$	$-3-3M$	0	$3+4M$	0	$6-12M$	--
	x_1	0	1	1	-1	0	1	0	2	--
	x_4	0	0	1	0	1	0	0	6	--
	a_2	0	0	-1	3	0	-3	1	12	4

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Iteration	Basis	Z	x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
3	Z	1	0	-3	0	0		$M1 + M18$		--
	x_1	0	1	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	6	9
	x_4	0	0	1	0	1	0	0	6	6
	x_3	0	0	$-\frac{1}{3}$	1	0	-1	$\frac{1}{3}$	4	--
4	Z	1	0	0	0	3		$M1 + M36$		--
	x_1	0	1	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	2	--
	x_2	0	0	1	0	1	0	0	6	--
	x_3	0	0	0	1	$\frac{1}{3}$	-1	$\frac{1}{3}$	6	--

It is found that, at iteration 4, optimality has reached. Optimal solution is $Z = 36$ with $x_1 = 2$ and $x_2 = 6$. The methodology explained above is known as *Big-M* method. Hope, reader has already understood the meaning of the terminology!

'Unbounded', 'Multiple' and 'Infeasible' solutions in the context of Simplex Method

As already discussed in lecture notes 2, a linear programming problem may have different type of solutions corresponding to different situations. Visual demonstration of these different types of situations was also discussed in the context of graphical method. Here, the same will be discussed in the context of Simplex method.

Unbounded solution

If at any iteration no departing variable can be found corresponding to entering variable, the value of the objective function can be increased indefinitely, i.e., the solution is unbounded.

Multiple (infinite) solutions

If in the final tableau, one of the non-basic variables has a coefficient 0 in the Z-row, it indicates that an alternative solution exists. This non-basic variable can be incorporated in the basis to obtain another optimal solution. Once two such optimal solutions are obtained, infinite number of optimal solutions can be obtained by taking a weighted sum of the two optimal solutions.

Consider the slightly revised above problem,

$$
\begin{array}{ll}
\text{Maximize} & Z = 3x_1 + 2x_2 \\
\text{subject to} & x_1 + x_2 \geq 2 \\
 & x_2 \leq 6 \\
 & 3x_1 + 2x_2 = 18 \\
 & x_1 , x_2 \geq 0
\end{array}
$$

Curious readers may find that the only modification is that the coefficient of x_2 is changed from 5 to 2 in the objective function. Thus the slope of the objective function and that of third constraint are now same. It may be recalled from lecture notes 2, that if the *Z line* is parallel to any side of the feasible region (i.e., one of the constraints) all the points lying on that side constitute optimal solutions (refer fig 3 in lecture notes 2). So, reader should be able to imagine graphically that the LPP is having infinite solutions. However, for this particular set of constraints, if the objective function is made parallel (with equal slope) to either the first constraint or the second constraint, it will not lead to multiple solutions. The reason is very simple and left for the reader to find out. As a hint, plot all the constraints and the objective function on an arithmetic paper.

Now, let us see how it can be found in the simplex tableau. Coming back to our problem, final tableau is shown as follows. Full problem is left to the reader as practice.

Final tableau:

Iteration	Basis	Z	x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
	Z	1	0	0	0	0		$M1 + M18$		--
3	x_1	0	1	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	6	9
	x_4	0	0	1	0	1	0	0	6	6
	x_3	0	0	$-\frac{1}{3}$	1	0	-1	$\frac{1}{3}$	4	--

Coefficient of non-basic variable x_2 is zero

As there is no negative coefficient in the Z-row the optimal is reached. The solution is $Z = 18$ with $x_1 = 6$ and $x_2 = 0$. However, the coefficient of non-basic variable x_2 is zero as shown in the final simplex tableau. So, another solution is possible by incorporating x_2 in the basis.

Based on the $\frac{b_r}{c_{rs}}$, x^4 will be the exiting variable. The next tableau will be as follows:

Iteration	Basis	Z	x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
	Z	1	0	0	0	0		$M1 + M18$		--
4	x_1	0	1	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	2	--
	x_2	0	0	1	0	1	0	0	6	6
	x_3	0	0	0	1	$\frac{1}{3}$	-1	$\frac{1}{3}$	6	18

Coefficient of non-basic variable x_4 is zero

Thus, another solution is obtained, which is $Z = 18$ with $x_1 = 2$ and $x_2 = 6$. Again, it may be noted that, the coefficient of non-basic variable x_4 is zero as shown in the tableau. If one more similar step is performed, same simplex tableau at iteration 3 will be obtained.

Thus, we have two sets of solutions as $\begin{matrix} \\ 0 \end{matrix}$ and $\begin{matrix} \\ 6 \end{matrix}$. Other optimal solutions will be obtained as $\beta \begin{matrix} 6 \\ 0 \end{matrix} + 1 - \beta \begin{matrix} 2 \\ 6 \end{matrix}$ where, $\beta \in 0,1$. For example, let $\beta = 0.4$, corresponding solution is $\begin{matrix} 3.6 \\ 3.6 \end{matrix}$, i.e., $x_1 = 3.6$ and $x_2 = 3.6$. Note that values of the objective function are not changed for different sets of solution; for all the cases $Z = 18$.

Infeasible solution

If in the final tableau, at least one of the artificial variables still exists in the basis, the solution is indefinite.

Reader may check this situation both graphically and in the context of Simplex method by considering following problem:

$$
\begin{array}{ll}
\text{Maximize} & Z = 3x_1 + 2x_2 \\
\text{subject to} & x_1 + x_2 \le 2 \\
 & 3x_1 + 2x_2 \ge 18 \\
 & x_1 , x_2 \ge 0
\end{array}
$$

Minimization versus maximization problems

As discussed earlier, standard form of LP problems consist of a maximizing objective function. Simplex method is described based on the standard form of LP problems, i.e., objective function is of maximization type. However, if the objective function is of minimization type, simplex method may still be applied with a small modification. The required modification can be done in either of following two ways.

1. The objective function is multiplied by -1 so as to keep the problem identical and ‘minimization’ problem becomes ‘maximization’. This is because of the fact that minimizing a function is equivalent to the maximization of its negative.
2. While selecting the entering nonbasic variable, the variable having the maximum coefficient among all the cost coefficients is to be entered. In such cases, optimal solution would be determined from the tableau having all the cost coefficients as non-positive (≤ 0)

Still one difficulty remains in the minimization problem. Generally the minimization problems consist of constraints with 'greater-than-equal-to' (≥) sign. For example, minimize the price (to compete in the market); however, the profit should cross a minimum threshold. Whenever the goal is to minimize some objective, lower bounded requirements play the leading role. Constraints with 'greater-than-equal-to' (≥) sign are obvious in practical situations.

To deal with the constraints with 'greater-than-equal-to' (≥) and = sign, *Big-M* method is to be followed as explained earlier.

UINT - IV

1 Data: Representation, Average, Spread

Data can be represented numerically or graphically in various ways. For instance, your daily newspaper may contain tables of stoke prces and money exchange rates, curves or bar of charts illustrating economical or political developments, or pie charts showing how your tax dollar is spent. And there are numerous other representations of data for special purposes.

In this section we discuss the use of standard representations of data in statistics.

We explain corresponding concepts and methods in terms of typical examples beginning with

$$89 \quad 84 \quad 87 \quad 81 \quad 89 \quad 86 \quad 91 \quad 90 \quad 78 \quad 89 \quad 87 \quad 99 \quad 83 \quad 89. \tag{1}$$

These are $n = 14$ measurements of the tensile strength of sheet steel in kg/mm^2 recorded in the order obtained and rounded to integer values. To see what is going on, we aort these data, that is, we order them by size

$$78 \quad 81 \quad 83 \quad 84 \quad 86 \quad 87 \quad 87 \quad 89 \quad 89 \quad 89 \quad 89 \quad 90 \quad 91 \quad 99. \tag{2}$$

Graphical Representation of Data

We shall now discuss standard graphical representations used in statistics for obtaining information on properties of data.

Stem and Leaf Plot

This is one of the simplest but most useful representations of data. For data (1) it is shown below.

Leaf unit =1.0

1	4	8
4	8	134
11	8	6779999
13	9	01
14	9	9

The numbers in (1) range from 78 to 99. We divide these numbers into 5 groups, $75-79, 80-84, 85-89, 90-94, 95-99$. The integers in the tens position of the groups are $7, 8, 8, 9, 9$. These form the stem. The first leaf is 8. The second leaf is 134(representing $81, 83, 84$), and so on.

The number of times a value occurs is called its absolute frequency. Thus 78 has absolute frequency 1, the value 89 has absolute frequency 4 etc. The column to the extreme left shows the cummulative absolute frequency, that is the sum of the absolute frequencies of the values up to the line of the leaf. Thus the number 4 in the second line on the left shows that (1) has 4 values up to and including 84. The number 11 in the next line shows that there are 11 values not exceeding 89, etc. Dividing the cummulative absolute frequencies by $n(=14)$ gives the cummulative relative frequencies.

Histogram

For large sets of data, histograms are better in displayng the distribution of data than stem-and-leaf plots.

Center and Spread of data: Median

As a center of the location of data values we can simply take the median, the data value that falls in the middle when the values are ordered. In (2) we have 14 values. The seventh of them is 87, the eighth is 89, and we the split the difference, obtaining the median 88.

Mean, Standard deviation, Variance

The average size of the data values can be measured in a more refined way by the mean

$$\overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}(x_1 + x_2 + \cdots + x_n). \tag{3}$$

This is the arithmetic mean of the data values, obtained by taking their sum and dividing by the data size n. Thus in (1),

$$\overline{x} = \frac{1}{14}(89 + 84 + \cdots + 89) = \frac{611}{7} \approx 87.3.$$

Similarly, the spread of the data values can be measured in a more refined way by the standard deviation s or by its square, the variance

$$s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \overline{x})^2 = \frac{1}{n-1}[(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2].$$

Thus to obtain the variance of the data, take the difference $x_j - \overline{x}$ of each data value from the mean, square it, take the sum of these n squares, and divide it by $n-1$. To get the standard deviation s, take the square root of s^2.

For example, using $\overline{x} = 611/7$, we get the data (1) the variance

$$s^2 = \frac{1}{13}[(89 - \frac{611}{7})^2 + (84 - \frac{611}{7})^2 + \cdots + (89 - \frac{611}{7})^2] = \frac{176}{7} \approx 25.14.$$

2 Experiments, Outcomes, Events

An experiment is a process of measurement or observation, in a labortory, in a factory, on the street, in nature or wherever; so “experiment” is used in a rather general sense.

Our interest is in experiments that involve “randomness”, chance effects, so that we can not predict a result exactly.

A “trail” is a single performance of an experiment. Its result is called an “outcomes” or a “sample point”. n trails then give a “sample” of “size n” consisting of n sample points.

The “sample space S” of an experiment is the set of all possible outcomes.

Examples are:

(1) Inspecting a lightbulb. $S = \{\text{Defective, Nondefective}\}$.

(2) Rolling a die. $S = \{1, 2, 3, 4, 5, 6\}$.

(3) Measuring tensile strength of wire. S the numbers in some interval.

(4) Measuring copper content of brass. S: 50% to 90%.

(5) Counting daily traffic accidents in New York. S the integers in some interval.

The subsets of S are called “events” and the outcomes “simple events”.

3 Probability

The “probability” of an event A in an experiment is supposed to measure how frequenty A is about to occur if we make any trials.

Defination 1. Probability

If the sample space S of an experiment consists of finitely many outcomes that are equally likely, then the probability $P(A)$ of an event A is

$$P(A) = \frac{\text{Number of points in } A}{\text{Number of points in } S}.$$

Thus in particular,

$$P(S) = 1.$$

Defination 2. Probability

Given a sample space S, with each event A of S there is associated a number $P(A)$, called the probability of A, such that the following axioms of probability are satisfied.

1. For every A in S,

$$0 \leqq P(A) \leqq 1.$$

2. The entire sample space S has the probability

$$P(S) = 1.$$

3. For mutually exclusive events A and $B(A \cap B = \emptyset)$ then

$$P(A \cup B) = P(A) + P(B).$$

4 Random variables, Probability Distributions

A probability distribution shows that the probabilities of events in an experment. The quantity that we observe in an experiment will be denoted by X and is called a random variable because the value it will assume in the next trial depends on chance, on randomness.

If we count (cars on the road, deaths by cancer, etc), we have a discrete random variable and distribution. If we measure (electric voltage, rainfall, hardness of steel), we have a continuous random variable and distribution.

In both the cases the distribution of X is determined by the distribution function

$$F(x) = P(X \leqq x); \tag{4}$$

this is the probability that X will assume any value not exceeding x. From (4) we obtain the fundamental formula for the probability corresponding to an interval $a < x \leqq b$,

$$P(a < X \leqq b) = F(b) - F(a).$$

Discrete random variabls and Distributions

A random variable X and its distribution are discrete if X assumes only finitely many or at most countably many values $x_1, x_2, \cdots$, called the probability values of X, with positive probabilities $p_1 = P(X = x_1), p_2 = P(X = x_2), \cdots$, whereas the probability of $P(X \in I)$ is zero for any interval I containing no possible value.

Obviously, the discrete distribution is also determined by the probability function $f(x)$ of X, defined by

$$f(x) = \begin{cases} p_j & \text{if } x = x_j, (j = 1, 2, \cdots) \\ 0 & \text{otherwise }. \end{cases}$$

From this we get the values of the distribution function $F(x)$ by takng sums,

$$F(x) = \sum_{x_j \leqq x} f(x_j) = \sum_{x_j \leqq x} p_j.$$

where for any given x we sum all the probabilities p_j for which x_j is smaller that or equal to that x.

For the probability corresponding to intervals we have

$$P(a < X \leqq b) = \sum_{a < x_j \leqq b} p_j.$$

This is the sum of all probabilities p_j for which x_j satisfies $a < x_j \leqq b$.

Note: $\sum_j p_j = 1$.

Continuous random variabls and Distributions

A random variable X and its distribution are of continuous type or, briefly, continuous, if its distribution function $F(x)$ can be given by an integral

$$F(x) = \int_{-\infty}^{x} f(v)dv,$$

whose integrand $f(x)$, called the density of the distribution.

Differentiating gives the relation of f to F as

$$f(x) = F'(x)$$

for every x at which $f(x)$ is continuous.

For the probability corresponding to an interval:

$$P(a < X \leqq b) = F(b) - F(a) = \int_a^b f(v)dv.$$

Note:
$$\int_{-\infty}^{\infty} f(v)dv = 1.$$

Example

Let the random variable X =Sum of the two numbers when two dice turn up. The probability function and the distribution function are as follows:

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$F(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	$\frac{36}{36}$

Example

Let X have the density function $f(x) = \begin{cases} 0.75(1-x^2) & \text{if } -1 \leqq x \leqq 1, \\ 0 \quad \text{otherwise .} \end{cases}$ Find the distribution function. Find the probability $P(-\frac{1}{2} \leqq X \leqq \frac{1}{2})$ and $P(-\frac{1}{4} \leqq X \leqq 2)$. Fnd x such that $P(X \leqq x) = 0.95$.

Solution:

From the defination, it is clear that $F(x) = 0$ if $x \leqq -1$,

$$F(x) = 0.75 \int_{-1}^{x} (1-v^2)dv = 0.5 + 0.75x - 0.25x^3, \quad -1 < x \leqq 1,$$

and $F(x) = 1$ for $x > 1$.

$$P(-\frac{1}{2} \leqq X \leqq \frac{1}{2}) = F(\frac{1}{2}) - F(-\frac{1}{2}) = 0.75 \int_{-1/2}^{1/2} (1-v^2)dv = 68.75\%$$

$$P(\frac{1}{4} \leqq X \leqq 2) = F(2) - F(\frac{1}{4}) = 0.75 \int_{1/4}^{2} (1-v^2)dv = 31.64\%$$

Finally,

$$P(X \leqq x) = F(x) = 0.5 + 0.75x - 0.25x^3 = 0.95.$$

Algebraic simplification gives $3x - x^3 = 1.8$. A solution is $x = 0.73$, approximately.

5 Mean and variance of a distribution

The mean μ and variance σ^2 of a random variable X and its distribution are the theoretical counterparts of the mean $\overline{x}$ and variance s^2 of a frequency distribution. The mean μ is defined by

$$\mu = \begin{cases} \sum_j x_j f(x_j) & \text{(Discrete distribution)}, \\ \int_{-\infty}^{\infty} x f(x) dx & \text{(Continuous distribution)}, \end{cases}$$

and the varance σ^2 is defined by

$$\sigma^2 = \begin{cases} \sum_j (x_j - \mu)^2 f(x_j) & \text{(Discrete distribution)}, \\ \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx & \text{(Continuous distribution)}, \end{cases}$$

σ is called the standard deviation of X and its distribution. f is the probability function (or probability mass function) or the density function, respectively, in discrete and continuous distribution.

6 Binomial Distribution

Consider a set of n independent trials (n being finite) in which the probability p of success in any trail is constant for each trial, then $q = 1 - p$, is the probability of failure in any trail.

A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by:

$$P(X = x) = p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x} & x = 0, 1, 2, \cdots, n; \quad q = 1 - p,, \\ 0 \quad \text{otherwise}, \end{cases}$$

The mean of the binomial distribution is

$$\mu = np$$

and the variance is

$$\sigma^2 = npq.$$

Example:

Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.

Solution: Here p= Probability of gettng a head $=\frac{1}{2}$

q= Probability of not gettng a head $=\frac{1}{2}$

$\therefore$ Probability of getting at least seven heads is given by :
$P(X \geq 7) = p(7) + p(8) + p(9) + p(10) = \frac{176}{1024}$.

7 Poisson Distribution

A random variable X is said to follow Poisson distribution if it assumes only non-negative values and its probability mass function is given by:

$$P(X = x) = p(x, \lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, 2, \cdots ; \quad \lambda > 0 \\ 0 \quad \text{otherwise,} & \end{cases}$$

Here λ is known as the parameter of the distribution.

Here the mean and varance of the Poisson distribution are each equal to λ. It can be proved that this distribution is obtained as limting case of the binomial distribution, if we let $p \to 0$ and $n \to \infty$ so that the mean $np = \lambda$ is a finite value.

Example

If on the average, 2 cars enter a certain parking lot per minute, what is the probability that during any given minute, 4 or more cars will enter the lot?

Solution:

To understand that the Poisson distribution is a model of the situation, we imagine the minute to be divided into very many short time intervals, let p be that probability that a car will enter the lot during any such short interval, and assume independence of the events that happens during those intervals.

Then we are dealing with binomial distributionwith very large n and very small p, which we can approximate by the Poisson distribution with $\lambda = np = 2$.

Thus the complementary event "3 cars or fewer enter the lot" has the probability:

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right) = 0.857.$$

$\therefore \quad P(4 \text{ or more cars will enter the lot}) = 1 - 0.857 = 0.143$

8 Normal Distribution

A random variable X is said to have a normal distribution with parameters μ (called mean) and σ^2 (called variance) if its probability density function (pdf) is given by the probability law:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0.$$

Remark: When a r.v. X is normally distributed with mean μ and standard deviation σ, it is customary to write X as distributed as $N(\mu, \sigma^2)$ and is expressed as $X \sim N(\mu, \sigma^2)$.

Distribution function $F(x)$

The normal distribution has the distribution function

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} exp\left[-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2\right] dv.$$

For the corresponding standardized normal distribution with mean 0 and standard deviation 1 we denote $F(x)$ by

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

Result 1

The distribution function $F(x)$ of the normal distribution with any μ and σ s related to the standardized distribution function $\Phi(z)$ by the formula

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Result 2

The probability that a normal random variable X with mean μ and standard deviation σ assume any value in an interval $a < x \leqq b$ is

$$P(a < x \leqq b) = F(b) - F(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

9 Regression Analysis

Regression analysis is a mathematical measure of the average relationship between two or more variables in terms of the original units of the data.

Linear regression

If the variables in a bivariate distribution are related, we will find the points in the scatter diagram will cluster arround some curve called the "curve of regression". If the curve is a straight line, it is called the line of regression and there is said to be linear regression between two variables, otherwise regression is said to be curvilinear.

The line of regression is the line which gives the best estimate to the value of one variable for any specific value of other variable. Thus the line of regression is the line of "best fit" and is obtained by the principle of least squares.

Let us suppose that in the bivariate distribution $(x_i, y_i); = 1, 2, \cdots, n$; Y is dependent variable and X is independent variable. Let the line of regression of Y on X be

$$y = a + bx.$$

The above equation represents a family of straight lines for different values of the arbitrary constants a and b. The problem is to determine a and b so that the line $y = a + bx$ is the line of "best fit".

Using Least square method, we get the line of regression of Y on X passes through the point $(\overline{x}, \overline{y})$ as

$$y - \overline{y} = k_1(x - \overline{x}),$$

where $\overline{x}$ and $\overline{y}$ are the means of the $x-$ and $y-$ values in our sample, and the slope k_1 is called the regression coefficient, is given by

$$k_1 = \frac{s_{xy}}{s_x^2},$$

with the "sample covariance" s_{xy} given by

$$s_{xy} = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \overline{x})(y_j - \overline{y}) = \frac{1}{n-1}\left[\sum_{j=1}^{n} x_j y_j - \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right)\left(\sum_{j=1}^{n} y_j\right)\right],$$

and the "sample variance of the x-values" s_x^2 is given by

$$s_x^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \overline{x})^2 = \frac{1}{n-1}\left[\sum_{j=1}^{n} x_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right)^2\right].$$

Example Obtain the line of regression of Y on X for the following data.

X :	65	66	67	67	68	69	70	72
Y :	67	68	65	68	72	72	69	71

Solution: Here $\overline{x} = 68, \overline{y} = 69, s_x = 2.12, s_y = 2.35, s_{xy} = 2.9892.$

$\therefore$ Equation of line of regression of Y on X is:

$$y - 69 = \frac{2.9892}{(2.12)^2}(x - 68) \qquad \Rightarrow y = 0.665x + 23.78$$

10 Correlation Analysis

Correlation analysis is concerned with the relation between X and Y in a two-dimensional random variable (X, Y). A sample consists of n ordered pairs of values $(x_1, y_1), \cdots,$ (x_n, y_n), we shall use the sample means $\overline{x}$ and $\overline{y}$, the sample variances s_x^2 and s_y^2 and the sample covariance s_{xy}.
The sample correction coefficient is

$$r = \frac{s_{xy}}{s_x s_y}.$$

Remarks:
1. The correction coefficient r satisfies $-1 \leqq r \leqq 1$, and $r = \pm 1$ if and only if the sample values lie on a straight line.
2. Two independent variables are uncorrelated.
3. Correlation coefficient is independent of change of origin and scale.

11 Tests of Significance

A very important aspect of the sampling theory is the study of the tests of significance, which enables us to decide on the basis of sample results, if
(i) the deviation between the observed sample statistic and the hypothetical parameter values.
(ii) the deviation between two independent sample statistics; is significant or might be attributed to chance or the fluctuations of sampling.

11.1 Null and Alternative hypopthesis

For applying the test of significance we first set up a hypopthesis- a definite statement about the population parameter. Such a hypothesis, which is usually a hypothesis of no difference, is called "null hypopthesis" and usually denoted by H_0.

Any hypopthesis which is complementary to the null hypopthesis is called an "alternative hypopthesis" and usually denoted by H_1. For example if we want to test the null hypopthesis that the population has a specifiedmean μ_0, (say), i.e., $H_0 : \mu = \mu_0$ then the alternative hypopthesis could be:

$$H_1 : \mu \neq \mu_0 \ (i.e., \mu > \mu_0 \ \text{ or } \ \mu < \mu_0) \qquad (ii) H_1 : \mu > \mu_0 \qquad (iii) H_1 : \mu < \mu_0$$

The alternative hypopthesis in (i) is known as a "two-tailed alternative" and the alternatives in (ii) and (iii) are known as "right-tailed" and "left-tailed alternatives" respectively.

11.2 Critical values or significant values

The value of test statistics which separates the critical (or rejection) region and the acceptance region is called the critical value or significant value. It depends upon:
(i) The level of significance used, and
(ii) The alternative hypopthesis, whether it is two-tailed or single-tailed.

11.3 Procedure for testing of hypothesis

We now summarize below the various steps in testing of a statistical hypopthesis in a systematic manner.
1. Null hypopthesis. Set up the null hypopthesis H_0.
2. Alternative hypopthesis. Set up the alternative hypopthesis H_1. This will enable us to decide whether we have to use a single-tailed test or right-tailed test.
3. Level of significance. Choose the appropriate level of significance (α) depending on the reliability of the estimates and permission risk. This is to decided before sample is drawn.
4. Test statistic. Compute the test statistic.
5. Conclusion. We compare the computed value in step 4 with significant value (tabulated value) at the given level of significance. If the calculated value in in modulus is less than tabulated value, then the null hypopthesis is accepted.

11.3.1 Test of significance for single proportion

If X is the number of successes in n independent trials with constant probability P of success for each trial, then set the statistic as

$$Z = \frac{p - P}{\sqrt{PQ/n}}, \ \text{ where } \ p = X/n, \ \ Q = 1 - P.$$

11.3.2 Test of significance for dfference of proportions

Suppose we want to compare two distinct populations with respect to the prevalence of a certain attribute, say A, among theirs members. Let X_1 and X_2 be the number of persons possessing the given attributes A n random samples of sizes n_1 and n_2 from two populations respectively. Then the sample proportions are given by : $p_1 = X_1/n_1$ and $p_2 = X_2/n_2$.

Under $H_0 : P_1 = P_2$, the test statistic for difference proportions is given by

$$z = \frac{p_1 - p_2}{\sqrt{\widehat{P}\widehat{Q}(\frac{1}{n_1} + \frac{1}{n_2})}}, \quad \text{where} \quad \widehat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}, \quad \widehat{Q} = 1 - \widehat{P}.$$

11.3.3 Test of significance for single mean

Here the statistic is

$$z = \frac{\overline{x} - \mu}{\sigma/\sqrt{n}},$$

where n is the sample size with mean $\overline{x}$ from a normal population with mean μ and standard deviation σ.

11.3.4 Test of significance for difference of means

Let $\overline{x_1}$ be the mean of a sample of size n_1 from a population with mean μ_1 and variance σ_1^2 and let $\overline{x_2}$ be the mean of an independent random sample of size n_2 from another population with mean μ_2 and variance σ_2^2.

Here the test statistic becomes

$$z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{(\sigma_1^2/n_1) + (\sigma_2^2/n_2)}}.$$

For more details about the theory, workout examples and questions, see the book:
"Fundamentals of Mathematical Statistics" by S.C. Gupta and V.K. Kapoor.