PARTIAL DIFFERENTIAL EQUATION

A differential equation containing terms as partial derivatives is called a partial differential equation (PDE). The order of a PDE is the order of highest

partial derivative. The dependent variable z depends on independent variables x and y.

p = $\frac{\partial z}{\partial x}$, q= $\frac{\partial z}{\partial y}$, r= $\frac{\partial^2 z}{\partial x^2}$, s= $\frac{\partial^2 z}{\partial x\,\partial y}$, t= $\frac{\partial^2 z}{\partial y^2}$

For example: q + px = x + y is a PDE of order 1

s + t = x^2 is a PDE of order 2

<u>Formation of PDE by eliminating arbitrary constant</u>:

For f(x,y,z,a,b) = 0 differentiating w.r.to x,y partially and eliminating constants a,b we get a PDE

Example 1: From the equation $x^2 + y^2 + z^2 = 1$ form a PDE by eliminating arbitrary constant.

Solution: $z^2 = 1 - x^2 - y^2$

Differentiating w.r.to x,y partially respectively we get

$$2z\frac{\partial z}{\partial x} = -2x \;\; and \; 2z\frac{\partial z}{\partial y} = -2y$$

p = $\frac{\partial z}{\partial x}$ = - x/z and q= $\frac{\partial z}{\partial y}$ = - y/z

z = - x/p = -y/ q

qx = py is required PDE

Example 2 From the equation x/2 + y/3 + z/4 = 1 form a PDE by eliminating arbitrary constant.

Solution:

Differentiating w.r.to x,y partially respectively we get

$$\frac{1}{2} + \frac{1}{4}\frac{\partial z}{\partial x} = 0 \;\; and \; \frac{1}{2} + \frac{1}{4}\frac{\partial z}{\partial y} = 0$$

$$\frac{1}{4}\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 0$$

p = $\frac{\partial z}{\partial x}$ = q = $\frac{\partial z}{\partial y}$

p = q is required PDE

<u>Formation of PDE by eliminating arbitrary function</u>

Let u= f(x,y,z), v= g(x,y,z) and φ(u,v) = 0

We shall eliminate φ and form a differential equation

Example 3 From the equation z = f(3x-y)+ g(3x+y) form a PDE by eliminating arbitrary function.

Solution:

Differentiating w.r.to x,y partially respectively we get

$$p = \frac{\partial z}{\partial x} = 3f'(3x-y) + 3g'(3x+y) \;\; and \;\; q = \frac{\partial z}{\partial y} = -f'(3x-y) + g'(3x+y)$$

$$r = \frac{\partial^2 z}{\partial x^2} = 9f''(3x-y) + 9g''(3x+y) \;\; and \;\; t = \frac{\partial^2 z}{\partial y^2} = f''(3x-y) + g''(3x+y)$$

From above equations we get r = 9t which is the required PDE.

11.1

An equation involving atleast one partial derivatives of a function of 2 or more independent variable is called PDE. A PDE is linear if it is of first degree in the dependent variable and its partial derivatives. If each term of such an equation contains either dependent variable or one of its derivatives the equation is called homogeneous.

Important Linear PDE of second order

$U_{tt} = c^2U_{xx}$ (One dimensional Wave equation)

$U_t = c^2U_{xx}$ (One dimensional Heat equation)

$U_{xx} + U_{yy} = 0$ (Two dimensional Laplace equation)

$U_{xx} + U_{yy} + U_{zz} = 0$ (Three dimensional Laplace equation)

$U_{xx} + U_{yy} = f(x,y)$ (Two dimensional Poisson equation)

PROBLEMS

1. Verify that $U = e^{-t}$ Sin 3x is a solution of heat equation.

Solution: $U_t = -e^{-t}$ Sin 3x and $U_{xx} = -9e^{-t}$ Sin 3x

$U_t = c^2U_{xx}$ (One dimensional heat equation) .... (1)

Putting the partial deivativers in equation (1) we get

$-e^{-t}$ Sin 3x = $-9c^2e^{-t}$ Sin 3x

Hence it is satisfied for $c^2 = 1/9$

One dimensional heat equation is satisfied for $c^2 = 1/9$. Hence U is a solution of heat equation.

2. Solve Uxy = -Uy

Solution: Put U_y = p then $\frac{\partial p}{\partial x} = -p$

$$\frac{\partial p}{p} = -\partial x$$

Integrating we get ln p = - x + ln c(y)

∂ U/ ∂ y = p = e^{-x} c(y)

∂ U = e^{-x} c(y) ∂ y

Integrating we get U = e^{-x} (y) φ(y) + D(x) where φ(y) = ∫c(y) ∂ y

11.2 Modeling: One dimensional Wave equation

We shall derive equation of small transverse vibration of an elastic string stretch to length L and then fixed at both ends.

Assumptions.

1. The string is elastic and does not have resistance to bending.
2. The mass of the string per unit length is constant.
3. Tension caused by stretching the string before fixing it is too large. So we can neglect action of gravitational force on the string.
4. The string performs a small transverse motion in vertical plane. So every particle of the string moves vertically.

Consider the forces acting on a small portion of the string. Tension is tangential to the curve of string at each point.Let T_1 and T_2 be tensions at end points. Since there is no motion in horizontal direction, horizontal components of tension are

T_1 Cos α= T_2 Cos β = T = Constant .... (1)
The vertical components of tension are - T_1 Sin α and T_2 Sin β of T_1 and T_2

By Newton's second law of motion, resultant force = mass x acceleration

$$T_2 \text{ Sin } \beta - T_1 \text{ Sin } \alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \text{ Sin } \beta}{T} - \frac{T_1 \text{ Sin } \alpha}{T} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2} \qquad \text{......} \quad \text{................(2)}$$

As tan β = $(\partial u/ \partial x)_x$ = Slope of the curve of string at x

tan α = $(\partial u/ \partial x)_{x+\Delta x}$ = Slope of the curve of string at x+ Δx

Hence from equation (2) $(\partial u/ \partial x)_{x+\Delta x} - (\partial u/ \partial x)_x = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$

$[(\partial u/ \partial x)_{x+\Delta x} - (\partial u/ \partial x)_x]/\Delta x = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$ Dividing both sides by Δx

Taking limit as $\Delta x \to 0$ we get

$\text{Lim}_{\Delta x \to 0} [(\partial u/ \partial x)_{x+\Delta x} - (\partial u/ \partial x)_x]/\Delta x = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

$$OR \quad \frac{\partial^2 u}{\partial t^2} = C^2\frac{\partial^2 u}{\partial x^2} \quad where\ C^2 = \frac{T}{\rho}$$

which is One dimensional Wave equation

11.3 Solution of One dimensional Wave equation (separation of variable method)

One dimensional wave equation is $u_{tt} = c^2 u_{xx}$ ....(1)

Boundary Condition u(0, t) = 0, u(L,t) = 0 ....(2)

Initial Condition u(x,0) = f(x) = initial deflection ....(3)

u_t (x, 0) = g(x) = initial velocity ....(4)

<u>**Step I**</u> Let u(x,t) = F(x) A(t)

Then $u_{tt} = F(x)\ddot{A}(t)$ and $u_{xx} = F''(x) A(t)$

Equation (1) becomes $F(x)\ddot{A}(t) = C^2F''(x) A(t)$

$\ddot{A}(t)/ [C^2 A(t)] = F''(x)/ F(x)$

L.H.S. involves function of t only and R.H.S. involves function of x only. Hence both expression must be equal to some constant k.

$\ddot{A}(t)/ [C^2 A(t)] = F''(x)/ F(x) = k =$ constant

$F''(x) - k F(x) = 0$ ---------------------(6)

$\ddot{A}(t) - C^2 kA(t) = 0$ ....(7)

Step II

We have to find solutions of F and G of equations (6) and (7) so that u satisfies equation(2) .

Hence $u(0,t) = F(0) A(t)=0$ and $u(L,t) = F(L) A(t)=0$

If A = 0 then u = 0 and we can not get a valid solution of deflection u.

Let A is non zero then F(0) = 0 and F(L) = 0 ....(8)

Three cases may arise.

Case I : K = 0

From eq (6) $F'' = 0$

Integrating we get F = ax + b

Using (8) we get a = 0, b = 0 Hence F = 0 and u =0 which is of no interest.

Case II : $K = \alpha^2$ (Positive)

From eq (6) $F'' - \alpha^2 F= 0$

Integrating we get $F = ae^{\alpha x} + be^{-\alpha x}$

Using (8) we get a = 0, b = 0 Hence F = 0 and u =0 which is of no interest.

Case III : $K = -p^2$ (Positive)

From eq (6) $F'' + p^2 F= 0$

Integrating we get F = C Cos px + B Sin px

Using (8) we get F(0) = C = 0, F(L) = B Sin pL = 0

Let $B \neq 0$ then Sin pL = 0 Hence $pL = n\pi$ and $p = n\pi/L$

Putting B=1 we get $F(x) = \text{Sin } n\pi x/L$ ....(9)

So $F_n(x) = \text{Sin } n\pi x/L$ where n=1,2,3, ... Thus we get infinitely many solutions satisfying equation (8).

Putting $k = -p^2$ in equation (7) we get $\ddot{A}(t) + p^2C^2 A(t) = 0$

$\ddot{A}(t) + (C^2 n^2\pi^2/L^2) A(t) = 0$

OR $\ddot{A}(t) + (\lambda n)^2 A(t) = 0$ where $\lambda n = cn\,\pi/L$

General Solution $A_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t$ ....(10)

Hence $u_n(x,t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin n\pi x/L$ for n=1,2,3...... ....(11)

Are solutions of equation (1) satisfying boundary condition (2).

These functions are called eigen functions and the values $\lambda n = cn\,\pi/L$ are calledeigen values or characteristic values of the vibrating string.

Step III

A single solution un(x,t) shall not satisfy initial Conditions (3) and (4). To get a solution that satisfies (3) and (4) we consider the series

$$u(x,t) = \Sigma\, u_n(x,t) = \Sigma\, (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin n\pi x/L \quad ....(12)$$

From equations (12) and (3) we get $u(x,0) = \Sigma\,(B_n \sin n\pi x/L) = f(x)$ ....(13)

Bn must be chosen so that u (x,0) must be a half range expansion of f(x)

i.e. $B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\ \sin\frac{n\pi x}{L}\,dx$ ....(14)

Differentiating (12) w.r.to t and using (4) we get

$\Sigma\,(B_n^* \lambda_n \sin n\pi x/L) = g(x)$

For equation (12) to satisfy (4) the coefficient Bn* should be chosen so that for t = 0, ut becomes Fourier Sine series of g(x)

$$B_n^* = \frac{2}{Cn\pi}\int_0^L g(x)\ \sin\frac{n\pi x}{L}\,dx$$

PROBLEMS

1. Find the defection u(x,t) of the vibrating string of length L=π, ends fixed, C=1, with zero initial velocity and initial deflection x(π-x)

Solution: Given length L=π, C=1, initial velocity g(x) = 0 .Hence Bn* = 0 and

$\lambda n = cn\,\pi/L = n$

The initial deflection f(x) = x(π-x)

$$B_n = \frac{2}{L}\int_0^L f(x)\ \sin\frac{n\pi x}{L}\,dx \qquad = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{(\pi x - x^2)}{n}\cos nx + \frac{(\pi - 2x)}{n^2}\sin nx - \frac{2}{n^3}\cos nx\right]_0^{\pi}$$

$$= \frac{4}{n^3\pi}\left[1 - \text{Cos } n\pi\right]$$

$B_1 = 8/\pi$, $B_2 = 0$, $B_3 = 8/27\pi$

The defection u(x,t) of the vibrating string

$u(x,t) = \Sigma (Bn \text{ Cos } \lambda n\, t + Bn^* \text{ Sin } \lambda n\, t) \text{ Sin } n\pi x/L$

$= \Sigma (Bn \text{ Cos } n\, t) \text{ Sin } nx$ (as $Bn^* = 0$ and $L = \pi$)

$= B_1 \text{ Cos } t \text{ Sin } x + B_2 \text{ Cos } 2t \text{ Sin } 2x + \ldots\ldots\ldots\ldots.$

$= (8/\pi) \text{ Cos } t \text{ Sin } x + (8/27\pi) \text{Cos } 3t \text{ Sin } 3x + \ldots\ldots\ldots\ldots.$

2. Using separation of variable solve the PDE $U_{xy} = U$

Solution: Let $U = F(x)\, G(Y)$ then $U_x = F'\, G$ and $U_{xy} = \partial U_x/\partial y = F'\, G^*$

Where $F' = \partial F/\partial x$ and $G^* = \partial G/\partial y$

Putting these partial derivatives the given PDE becomes $F'\, G^* = F\, G$

By separation of variables we get $F'/F = G/G^* = k$ = Constant

(Since L.H.S. is a function of x and R.H.S. is a function of y)

$F'/F = k$ and $G/G^* = k$

$\partial F/F = k\partial x$ and $\partial G/G = \partial y/k$

Integrating both sides of these equations we get

$\ln F = kx + \ln C$ and $\ln G = y/k + \ln D$

$F = C\, e^{kx}$ and $G = D\, e^{y/k}$

$U = F\, G = C\, D\, e^{kx + y/k}$

11.4 **D ALEMBERT'S SOLUTION OF WAVE EQUATION**

One dimensional wave equation is $u_{tt} = c^2 u_{xx}$ ....(1)

We have to transform equation (1) by using new independent variables $v = x + ct$ and $z = x - ct$

$u = u(x,t)$ will become a function of v and z.

The partial derivatives are $\partial v/\partial x = 1 = \partial z/\partial x$, $\partial v/\partial t = c$ and $\partial z/\partial t = -c$ ....(2)

Using chain rule for function of several variables we get $u_x = u_v v_x + u_z z_x = u_v + u_z$

$u_{xx} = (\partial/\partial x)(u_v + u_z)$

$$= \frac{\partial}{\partial v}(u_v)\frac{\partial v}{\partial x} + \frac{\partial}{\partial z}(u_v)\frac{\partial z}{\partial x} + \frac{\partial}{\partial v}(u_z)\frac{\partial v}{\partial x} + \frac{\partial}{\partial z}(u_z)\frac{\partial z}{\partial x} = u_{vv} + u_{vz} + u_{vz} + u_{zz} = u_{vv} + 2u_{vz} + u_{zz}$$

Hence $u_{xx} = u_{vv} + 2u_{vz} + u_{zz}$ ....(3)

Similarly $u_t = u_v v_t + u_z z_t = cu_v - c u_z$

$u_{tt} = (\partial / \partial t)(cu_v - c u_z) = c(\partial / \partial t)u_v - c(\partial / \partial t) u_z$

$$= c\frac{\partial}{\partial v}(u_v)\frac{\partial v}{\partial t} + c\frac{\partial}{\partial z}(u_v)\frac{\partial z}{\partial t} - c\frac{\partial}{\partial v}(u_z)\frac{\partial v}{\partial t} - c\frac{\partial}{\partial z}(u_z)\frac{\partial z}{\partial t} = c^2 u_{vv} - c^2 u_{vz} - c^2 u_{vz} + c^2 u_{zz}$$

$u_{tt} = c^2(u_{vv} - 2u_{vz} + u_{zz})$ ....(4)

Using (3) and (4) in equation (1) we get $c^2(u_{vv} - 2u_{vz} + u_{zz}) = c^2(u_{vv} + 2u_{vz} + u_{zz})$

OR $-2u_{vz} = 2u_{vz}$ Hence $u_{vz} = 0$

$u_v = c(v)$

$u = \phi(v) + \psi(z) = \phi(x + ct) + \psi(x - ct)$

This is D Alemberts solution of wave equation where $\phi(v) = \int c(v)\, \partial v$

TYPES AND NORMAL FORM OF LINEAR PDE:

An equation of the form

A Uxx + 2B Uxy+ C Uyy = F(x,y,U,Ux,Uy) is said to be

elliptic if $AC - B^2 > 0$

parabolic if $AC - B^2 = 0$ and hyperbolic if $AC - B^2 < 0$

For parabolic equations the transform v= x, z = ψ(x, y) is used to transform to normal form

For hyperbolic equations the transform v=ϕ (x, y), z = ψ(x, y) is used to transform to normal form

Where ϕ = constant and ψ = constant are solutions of equation $Ay'^2 - 2By' + C = 0$

PROBLEMS

1. Given $f(x) = k(x - x^2)$, L=1, k =0.01, g(x)= 0 Find the deflection of the string.
   Solution: $f(x) = k(x - x^2)$
   $f(x + ct) = k[(x + ct) - (x + ct)^2]$ and $f(x - ct) = k[(x - ct) - (x - ct)^2]$

The deflection of the string is $u(x,t) = [f(x + ct) + f(x - ct)]/2$

$= k[x + ct - (x + ct)^2 + x - ct - (x - ct)^2]/2$
$= 0.01[x - x^2 - c^2t^2]$

2. Transform the PDE $4u_{xx} - u_{yy} = 0$ to normal form and solve

Solution : $4u_{xx} - u_{yy} = 0$ ....(1)

Here A=4, B = 0 and C= -1, hence $AC - B^2 = -4 < 0$

Given equation is a hyperbolic type equation.

From the equation $Ay'^2 - 2By' + C = 0$ we have $4y'^2 - 1 = 0$

Solving we get $x + 2y = c_1$ and $x - 2y = c_2$

We have to transform equation (1) by using new independent variables v = x + 2y and z = x-2y

u = u(x,t) will become a function of v and z.

The partial derivatives are $\partial v/\partial x = 1 = \partial z/\partial x$, $\partial v/\partial y = 2$ and $\partial z/\partial y = -2$ ....(2)

Using chain rule for function of several variables we get $u_x = u_v v_x + u_z z_x = u_v + u_z$

$u_{xx} = (\partial/\partial x)(u_v + u_z)$

$$= \frac{\partial}{\partial v}(u_v)\frac{\partial v}{\partial x} + \frac{\partial}{\partial z}(u_v)\frac{\partial z}{\partial x} + \frac{\partial}{\partial v}(u_z)\frac{\partial v}{\partial x} + \frac{\partial}{\partial z}(u_z)\frac{\partial z}{\partial x} = u_{vv} + u_{vz} + u_{vz} + u_{zz} = u_{vv} + 2u_{vz} + u_{zz}$$

Hence $u_{xx} = u_{vv} + 2u_{vz} + u_{zz}$ ....(3)

Similarly $u_y = u_v v_y + u_z z_y = 2u_v - 2u_z$

$u_{yy} = (\partial/\partial y)(cu_v - c u_z) = 2(\partial/\partial y)u_v - 2(\partial/\partial y)u_z$

$$= 2\frac{\partial}{\partial v}(u_v)\frac{\partial v}{\partial y} + 2\frac{\partial}{\partial z}(u_v)\frac{\partial z}{\partial y} - 2\frac{\partial}{\partial v}(u_z)\frac{\partial v}{\partial y} - 2\frac{\partial}{\partial z}(u_z)\frac{\partial z}{\partial y} = 4u_{vv} - 4u_{vz} - 4u_{vz} + 4u_{zz}$$

$u_{yy} = 4(u_{vv} - 2u_{vz} + u_{zz})$ ....(4)

Using (3) and (4) in equation (1) we get $4(u_{vv} - 2u_{vz} + u_{zz}) = 4(u_{vv} + 2u_{vz} + u_{zz})$

OR $-2u_{vz} = 2u_{vz}$ Hence $u_{vz} = 0$

$u_v = c(v)$

$u = \phi(v) + \psi(z) = \phi(x+2y) + \psi(x-2y)$

This is D Alemberts solution of wave equation where $\phi(v) = \int c(v)\,\partial v$

11.5 **Solution of One dimensional Heat equation (separation of variable method)**

One dimensional wave equation is $u_t = c^2 u_{xx}$ ....(1)

Boundary Condition u(0, t) = 0, u(L,t) = 0 ....(2)

Initial Condition u(x,0) = f(x) = initial temperature ....(3)

Step I Let u(x,t) = F(x) G(t) ....(4)

Then $u_t = F(x)\,G^*(t)$ and $u_{xx} = F''(x)\,G(t)$ where $F' = \partial F/\partial x$ and $G^* = \partial G/\partial t$

Equation (1) becomes $F(x)\, G^*(t) = C^2 F''(x)\, G(t)$

$G^*(t)/[C^2 G(t)] = F''(x)/F(x)$ ....(5)

L.H.S. involves function of t only and R.H.S. involves function of x only. Hence both expression must be equal to some constant k.

$G^*(t)/[C^2 G(t)] = F''(x)/F(x) = k = \text{constant}$

$F''(x) - k\, F(x) = 0$ ---------------------(6)

$G^*(t) - C^2 kG(t) = 0$ ....(7)

Step II

We have to find solutions of F and G of equations (6) and (7) so that u satisfies equation(2) .

Hence $u(0,t) = F(0)\, G(t)=0$ and $u(L,t) = F(L)\, G(t)=0$

If G = 0 then u = 0 and we can not get a valid solution of deflection u.

Let G is non zero then $F(0) = 0$ and $F(L) = 0$ ....(8)

Three cases may arise.

Case I : K = 0

From eq (6) $F'' = 0$

Integrating we get $F = ax + b$

Using (8) we get a = 0, b = 0 Hence F = 0 and u =0 which is of no interest.

Case II : $K = \alpha^2$ (Positive)

From eq (6) $F'' - \alpha^2 F= 0$

Integrating we get $F = ae^{\alpha x} + be^{-\alpha x}$

Using (8) we get a = 0, b = 0 Hence F = 0 and u =0 which is of no interest.

Case III : $K = -p^2$ (Positive)

From eq (6) $F'' + p^2 F= 0$

Integrating we get $F = A \text{ Cos } px + B \text{ Sin } px$

Using (8) we get $F(0) = A = 0$, $F(L) = B \text{ Sin } pL = 0$

Let $B \neq 0$ then $\text{Sin } pL = 0$ Hence $pL = n\pi$ and $p = n\pi/L$

Putting B=1 we get $F(x) = \text{Sin } n\pi x/L$ ....(9)

So $F_n(x) = \text{Sin } n\pi x/L$ where n=1,2,3, ... Thus we get infinitely many solutions satisfying equation (8).

Putting $k = -p^2$ in equation (7) we get $G^*(t) + p^2C^2 A(t) = 0$

$G^*(t) + (C^2 n^2\pi^2/L^2)\, G(t) = 0$

OR $G^*(t) + (\lambda n)^2\, G(t) = 0$ where $\lambda n = cn\,\pi/L$

General Solution $G_n(t) = B_n e^{-\lambda n2\, t}$ ....(10)

Hence $u_n(x,t) = B_n \sin n\pi x/L \;\; e^{-\lambda n2\, t}$ for n=1,2,3...... ....(11)

Are solutions of equation (1) satisfying boundary condition (2).

Step III

A single solution un(x,t) shall not satisfy initial Conditions (3) and (4). To get a solution that satisfies (3) and (4) we consider the series

$$u(x,t) = \Sigma\, u_n(x,t) = \Sigma\, B_n \sin n\pi x/L \;\; e^{-\lambda n2\, t} \qquad ....(12)$$

From equations (12) and (3) we get $u(x,0) = \Sigma\,(B_n \sin n\pi x/L) = f(x)$ ....(13)

Bn must be chosen so that u (x,0) must be a half range expansion of f(x)

i.e. $B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\ \sin\frac{n\pi x}{L}\,dx$ ....(14)

PROBLEMS

1. Find the temperature u(x,t) in a bar of length L= 10 cm, c=1,constant cross section area, which is perfectly insulated laterally and ends are kept at 0^0C, the initial temperature is x(10-x)

Solution: Given length L=10

$\lambda n = cn\,\pi/L = n\,\pi/10$

The initial deflection f(x) = x(10-x)

$$B_n = \frac{2}{10}\int_0^{10} f(x)\ \sin\frac{n\pi x}{10}\,dx \qquad = \frac{1}{5}\int_0^{\pi} (10x - x^2)\,\frac{\sin n\pi x}{10}\,dx$$

$$= \frac{1}{5}\left[-\frac{(10x-x^2)}{n}\cos nx + \frac{(10-2x)}{n^2}\sin nx - \frac{2}{n^3}\cos nx\right]_0^{\pi}$$

$$= \frac{400}{n^3\pi^3}\left[1-\cos n\pi\right]$$

$B_1 = 800/\pi^3$, $B_2 = 0$, $B_3 = 800/27\pi^3$

The temp u(x,t) of the bar

$u(x,t) = \Sigma\, B_n (\sin n\pi x/L)\, e^{-\lambda n2\, t}$

$= B_1 (\sin \pi x/10)\, e^{-\pi 2/100} + B_2 (\sin 3\pi x/10)\, e^{4(-0.017\pi 2t)}+$.................

$= (800/\pi^3)\sin \pi x/10\ e^{-0.017\pi 2t} + (800/27\pi^3)\sin 3\pi x/10\ e^{9(\pi 2/100)}+$.................

Insulated ends(Adiabatic Boundary Conditions)

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx$$

$$An = \frac{2}{L}\int_0^L f(x)\ \mathrm{Cos}\ \frac{n\pi x}{L}\, dx$$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \mathrm{Cos}\frac{n\pi x}{L}\ e - \lambda_n^2 t$$

2. Find the temperature u(x,t) in a bar of length L=π , c=1, which is perfectly insulated laterally and also ends are insulated, the initial temperature is x

Solution: Given length L=π

λn = cn π/L = n

The initial deflection f(x) = x

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} x dx = \pi/2$$

$$An = \frac{2}{L}\int_0^L f(x)\mathrm{Cos}\frac{n\pi x}{L}\, dx = \frac{2}{\pi}\int_0^{\pi} x\mathrm{Cos}nx\, dx = \frac{2}{\pi}\left[\frac{x\ \mathrm{Sin}\ nx}{n} + \frac{\mathrm{Cos}nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}\left(Cosn\pi - 1\right)$$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \mathrm{Cos}\frac{n\pi x}{L}\ e^{-\lambda_n^2 t}$$

$$= A_0 + A_1 \mathrm{Cos}\ \frac{\pi x}{L}\ e^{-\lambda_1^2 t} + A_2 \mathrm{Cos}\ \frac{2\pi x}{L}\ e^{-\lambda_2^2 t} + \ldots\ldots\ldots\ldots$$

$$= \pi/2 - 4/\pi\left[e^{-t}Cos\ x + \frac{e^{-9t}Cos\ 3x}{9} + \ldots\ldots\ldots\ldots\right]$$

UNIT-II

11.8 RECTANGULAR MEMBRANE

Two dimensional wave equation is $u_{tt} = c^2 (u_{xx} + u_{yy})$ ....(1)

Boundary Condition $u(x,y,t) = 0$ on the boundary of the membrane for all $t \geq 0$ ....(2)

Initial Conditions: $u(x,y,0) = f(x, y)$ = initial deflection ....(3)

$u_t(x, y,0) = g(x, y)$ = initial velocity ....(4)

Step I Let $u(x,y,t) = F(x, y) A(t)$

Then $u_{tt} = F(x, y) \ddot{A}(t)$ and $u_{xx} = F_{xx} A(t)$, $u_{yy} = F_{yy} A(t)$

Equation (1) becomes $F(x,y) \ddot{A}(t) = C^2(F_{xx} + F_{yy})A(t)$

$\ddot{A}(t)/ [C^2 A(t)] = (F_{xx} + F_{yy})/F$ ....(5)

L.H.S. involves function of t only and R.H.S. involves function of x only. Hence both expressions must be equal to some constant D.

For $D \geq 0$, as F= 0, hence u = o and we can not get solution.

For $D < 0$ let $D = - v^2$ (negative)

$\ddot{A}(t)/ [C^2 A(t)] = (F_{xx} + F_{yy})/F = - v^2$ = constant

$\ddot{A}(t) + v^2C^2A(t) = 0$

or $\ddot{A}(t) + \lambda^2A(t) = 0$ where $\lambda = c\, v$ ....(6)

$F_{xx} + F_{yy} + v^2 F = 0$ ....(7)

In equation (7) two variables x and y are present and we want to separate them.

Let $F(x,y) = H(x)Q(y)$ ....(8)

Then from equation (7) $Q\dfrac{d^2H}{dx^2} = -H\dfrac{d^2Q}{dy^2} - \nu^2 HQ = -H\left[\dfrac{d^2Q}{dy^2} + \nu^2 Q\right]$

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left[\frac{d^2Q}{dy^2} + \nu^2 Q\right]$$

L.H.S. is a function of x only and R.H.S. is a function of y only. Hence the expressions on both sides equal to a constant k. As negative value of constant leads to solution let the constant be $-k^2$ then,

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left[\frac{d^2Q}{dy^2} + \nu^2 Q\right] = -k^2$$

$$\frac{d^2H}{dx^2} + k^2 H = 0$$ ....(9)

$$\frac{d^2Q}{dy^2} + p^2Q = 0 \, where \quad p^2 = \nu^2 - k^2 \qquad \text{......................(10)}$$

Step II

General solution of equations (9) and (10) are

H(x) = A Cos kx + B Sin kx

Q(y) = C Cos py + D Sin py where A,B,C and D are constants.

From equations (5) and (2) we have F = HQ = 0 on the boundary.

Hence x=0,x=a, y=0, y=b implies H(0) = 0, H(a) = 0, Q(0) = 0, Q(b) = 0

Now H(0) = 0 implies A = 0

H(a) = 0 implies B Sin ka = 0

Assume B ≠ 0 then Sin ka = 0 (Because if B= 0 then H = 0 and hence F =0)

ka =mπ or k= mπ/a, m is integer

Again Q(0) = 0 implies C = 0

Q(b) = 0 implies D Sin pb = 0

Assume D ≠ 0 then Sin pb = 0 (Because if D= 0 then Q = 0 and hence F =0)

pb =nπ or p=nπ/b, n is integer

Thus H_m (x) = Sin mπx/a , m= 1,2,......

Q_n (y) = Sin nπy/b , n= 1,2,......

F_{mn} (x, y) = Sin mπx/a Sin nπy/b , m= 1,2,...... and n= 1,2,...... are solutions of equation (7) which are zero on the boundary of the membrane.

λ=cv== $c\sqrt{k^2 + p^2}$

$$\lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$ m= 1,2,...... and n= 1,2,......

The numbers λ_{mn} are called eigen values or characteristic values.

The general solution of (6) is

Amn (t) = B_{mn} Cos λ_{mn} t+ B_{mn}*Sin λ_{mn}t

Hence u_{mn}(x,y,t) = (B_{mn} Cos λ_{mn} t+ B_{mn}*Sin λ_{mn}t) Sin (mπx/a) Sin (nπy/b)......................(13)

Step III

We consider the series

$$u\ (x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn} Cos\lambda_{mn}t + B_{mn}^{*}\ Sin\lambda_{mn}t\right] Sin\frac{m\pi x}{a} Sin\frac{n\pi y}{b} \quad \text{................(14)}$$

From equations (14) and (3) we get

$$u\ (x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn}\right] Sin\frac{m\pi x}{a} Sin\frac{n\pi y}{b} = f(x,y) \quad \text{...............(15)}$$

This series is called a double Fourier series.

To find the Fourier coefficient B_{mn} , we put $K_m(y) = \sum_{n=1}^{\infty} B_{mn} Sin\frac{n\pi y}{b}$ $in\ equation\ (15)$

$$we\ \ get\ \ f(x,y) = \sum_{m=1}^{\infty} Km(y) Sin\frac{m\pi x}{a}$$

The coefficient $K_m(y) = \frac{2}{a}\int_0^a \mathrm{f}(\mathrm{x},\ \mathrm{y})\ \mathrm{Sin}\ \frac{\mathrm{m}\pi \mathrm{x}}{a} dx$

$$Hence\quad B_{mn} = \frac{2}{b}\int_0^b \mathrm{K}_{\mathrm{m}}\ (\mathrm{y})\ \mathrm{Sin}\ \frac{\mathrm{n}\pi \mathrm{y}}{b} dy$$

$$= \frac{4}{ab}\int_0^b\int_0^a \mathrm{f}(\mathrm{x},\ \mathrm{y})\ \ \mathrm{Sin}\ \frac{\mathrm{m}\pi x}{a} \mathrm{Sin}\ \frac{\mathrm{n}\pi \mathrm{y}}{b} dx\, dy$$

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\lambda_{mn} B_{mn}^{*}\ \right] Sin\frac{m\pi x}{a} Sin\frac{n\pi y}{b} = g(x,y)$$

$$B_{mn}^{*} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a \mathrm{g}(\mathrm{x},\mathrm{y})\ \ \mathrm{Sin}\ \frac{\mathrm{m}\pi x}{a} \mathrm{Sin}\ \frac{\mathrm{n}\pi y}{b} dxdy$$ where m= 1,2,...... and n= 1,2,......

PROBLEMS

1. Find the deflection u(x,y,t) of the square membrane a=1, b=1 and c=1 if the initial velocity is zero and initial deflection is $k(x-x^2)(y-y^2)$

Solution:

Given a=1, b=1 and c=1 . Hence we have

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} = \pi\sqrt{m^2+n^2}$$

The initial velocity g(x,y) is zero . Hence $B_{mn}^* = 0$

The initial deflection f(x,y) = $k(x-x^2)(y-y^2)$

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,\ y)\ \text{Sin}\frac{m\pi x}{a}\text{Sin}\frac{n\pi y}{b}\,dx\,dy = 4\int_0^1\int_0^1 k(x - x^2)(y - y^2)\ \text{Sin}\ m\pi x\,\text{Sin}\ n\pi y\,dx\,dy$$

$$= 4k\int_0^1\int_0^1\left[(y - y^2)\text{Sin}\ n\pi y\right](x - x^2)\text{Sin}\ m\pi x\,dx\,dy \qquad ....(1)$$

Now $\int_0^1\left[(y - y^2)\text{Sin}\ n\pi y\right]$dy

$$= -\left[(y - y^2)\frac{\text{Cos}\,n\pi y}{n\pi}\right]_0^1 + \int_0^1 (1 - 2y)\frac{\text{Cos}\,n\pi y}{n\pi}dy$$

$$= 0 - 0 + \frac{1}{n\pi}\left[\left[(1 - 2y)\frac{\text{Sin}\ \ n\pi y}{n\pi}\right]_0^1 - \int_0^1 (-2)\frac{\text{Sin}\ \ n\pi y}{n\pi}dy\right]$$

$$= \frac{1}{n\pi}\left[0 - 0 - \frac{2\text{Cos}\ \ n\pi y}{n^2\pi^2}\right]_0^1 = \frac{2}{n^3\pi^3}\left[1 - \text{Cos}\ \ n\pi\right] \qquad ....(2)$$

Putting this in equation (1)

$$Bmn = 4k\int_0^1 \frac{2}{n^3\pi^3}\left(1 - \text{Cos}\ n\pi\right)(x - x^2)\text{Sin}\ m\pi x\,dx$$

$$= \frac{8k}{n^3\pi^3}\left(1 - \text{Cos}\ n\pi\right)\int_0^1 (x - x^2)\text{Sin}\ m\pi x\,dx$$

$$= \frac{8k}{n^3\pi^3}\left(1 - \text{Cos}\,n\pi\right)\left[-(x - x^2)\frac{\text{Cos}\,m\pi x}{m\pi} + (1 - 2x)\frac{\text{Sin}\ m\pi x}{m^2\pi^2} - \frac{2}{m^3\pi^3}\left(\text{Cos}\,m\pi x\right)\right]_0^1$$

$$= \frac{8k}{n^3\pi^3}\left(1 - \text{Cos}\,n\pi\right)\left[\frac{2}{m^3\pi^3}\left(1 - \text{Cos}\,m\pi\right)\right]$$

$$= \frac{16k}{m^3n^3\pi^6}\left(1 - \text{Cos}\ n\pi\right)\left(1 - \text{Cos}\ m\pi\right)$$

$$Deflection\ u\ (x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn}Cos\lambda_{mn}t + B_{mn}^{*}\ \ Sin\lambda_{mn}t\right]Sin\frac{m\pi x}{a}Sin\frac{n\pi y}{b}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn}Cos\lambda_{mn}t\right]Sin\ m\pi x\,Sin\ n\pi y$$

as Bmn* = 0 and a=1, b=1

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{16k}{m^3n^3\pi^6}\left(1 - Cos\,n\pi\right)\left(1 - Cos\,m\pi\right)Cos\,\pi\sqrt{m^2 + n^2}\,t\,Sin\ m\pi x\,Sin\ n\pi y$$

2. Find the double Fourier series of f(x,y)= xy, 0<x<π and 0<y<π,

Solution:

Here a= π , b= π, f(x,y) = xy . Hence we have

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,\ y)\ \text{Sin}\,\frac{m\pi x}{a}\,\text{Sin}\,\frac{n\pi y}{b}\,dx\,dy = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi xy\ \text{Sin}\,mx\ \text{Sin}\,ny\,dx\,dy$$

$$= \frac{4}{\pi^2}\int_0^\pi\left[\frac{-y}{n}\text{Cos}\,ny + \frac{\text{Sin}\,ny}{n^2}\right]_0^\pi x\,\text{Sin}\,mx\,dx \qquad \text{..................(1)}$$

$$= \frac{4}{\pi^2}\int_0^\pi\left[\frac{-\pi}{n}\text{Cos}\,n\pi\right]x\,\text{Sin}\,mx\,dx = -\frac{4}{n\pi}\text{Cos}\,n\pi\int_0^\pi x\,\text{Sin}\,mx\,dx$$

$$= -\frac{4}{n\pi}Cosn\pi\left[\frac{-x}{m}\text{Cos}\,mx + \frac{\text{Sin}\,mx}{m^2}\right]_0^\pi = \frac{4}{mn}Cosn\pi\,Cosm\pi$$

................(2)

The double Fourier series is

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[B_{mn}]Sin\frac{m\pi x}{a}\,Sin\frac{n\pi y}{b} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[B_{mn}]SinmxSinny$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{4}{mn}Cosm\pi Cosn\pi\right]SinmxSinny$$

11.9 **LAPLACIAN IN POLAR COORDINATE**

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ \ (Laplace\ equation)$$

..............(1)

To convert Laplace equation (1) into polar form we put x = r Cos θ ....(2)

and y = r Sin θ ....(3)

Squaring and adding equations (2) and (3) we have $x^2+y^2 = r^2$ ....(4)

Dividing equations (3) by equation (2), tan θ = y/x , hence θ = $\tan^{-1}$ (y/x) ....(5)

Differentiating equation (2) partially w.r.to x we get $2x = 2r\frac{\partial r}{\partial x}$

Differentiating equation (2) partially w.r.to y we get $2y = 2r\frac{\partial r}{\partial y}$

Hence $r_x = \frac{\partial r}{\partial x} = \frac{x}{r}\ \ and\ r_y = \frac{\partial r}{\partial y} = \frac{y}{r}$ ....(6)

Again differentiating equation (6) partially w.r.to x and y respectively we get

$$r_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial r}{\partial x}\right] = \frac{r - x r_x}{r^2} = \frac{r - x(x/r)}{r^2} = \frac{r^2 - x^2}{r^3} = \frac{x^2 + y^2 - x^2}{r^3} = \frac{y^2}{r^3}$$

$$r_{yy} = \frac{\partial}{\partial y}\left[\frac{\partial r}{\partial y}\right] = \frac{r - y r_y}{r^2} = \frac{r - y(y/r)}{r^2} = \frac{r^2 - y^2}{r^3} = \frac{x^2 + y^2 - y^2}{r^3} = \frac{x^2}{r^3}$$

Differentiating equation (5) partially w.r.to x we get

$$\theta_x = \frac{\partial \theta}{\partial x} = \frac{1}{1 + y^2/x^2} \frac{\partial}{\partial x}\left[\frac{y}{x}\right] = \frac{1}{1 + y^2/x^2} y\left[\frac{-1}{x^2}\right] = \frac{-y}{x^2 + y^2} = \frac{-y}{r^2} \quad \text{.............(7)}$$

Differentiating equation (5) partially w.r.to y we get

$$\theta_y = \frac{\partial \theta}{\partial y} = \frac{1}{1 + y^2/x^2} \frac{\partial}{\partial y}\left[\frac{y}{x}\right] = \frac{1}{1 + y^2/x^2}\left[\frac{1}{x}\right] = \frac{x}{x^2 + y^2} = \frac{x}{r^2} \quad \text{..............(8)}$$

Again differentiating equation (7) partially w.r.to x we get

$$\theta_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial \theta}{\partial x}\right] = -y\frac{\partial}{\partial x}\left[\frac{1}{r^2}\right] = -y\left[\frac{-2}{r^3}\frac{\partial r}{\partial x}\right] = -y\left[\frac{-2}{r^3}\frac{x}{r}\right] = \frac{2xy}{r^4}$$

Again differentiating equation (8) partially w.r.to y we get

$$\theta_{yy} = \frac{\partial}{\partial y}\left[\frac{\partial \theta}{\partial y}\right] = x\frac{\partial}{\partial y}\left[\frac{1}{r^2}\right] = x\left[\frac{-2}{r^3}\frac{\partial r}{\partial y}\right] = x\left[\frac{-2}{r^3}\frac{y}{r}\right] = \frac{-2xy}{r^4}$$

Using chain rule for function of several variables we get

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = u_r r_x + u_\theta \theta_x$$

$$u_{xx} == \frac{\partial}{\partial x}\left(u_r r_x + u_\theta \theta_x\right) = \frac{\partial}{\partial x}\left(u_r r_x\right) + \frac{\partial}{\partial x}\left(u_\theta \theta_x\right)$$

$$= u_r \frac{\partial}{\partial x}(r_x) + r_x \frac{\partial}{\partial x}(u_r) + u_\theta \frac{\partial}{\partial x}(\theta_x) + \theta_x \frac{\partial}{\partial x}(u_\theta)$$

$$= u_r r_{xx} + r_x\left[\frac{\partial}{\partial r}(u_r)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}(u_r)\frac{\partial \theta}{\partial x}\right] + u_\theta \theta_{xx} + \theta_x\left[\frac{\partial}{\partial r}(u_\theta)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}(u_\theta)\frac{\partial \theta}{\partial x}\right]$$

$$= u_r\left[\frac{y^2}{r^3}\right] + r_x\left[r_x u_{rr} + (\theta_x u_{r\theta})\right] + u_\theta\left[\frac{2xy}{r^4}\right] + \theta_x\left[r_x u_{r\theta} + (\theta_x u_{\theta\theta})\right]$$

$$= \frac{y^2}{r^3}u_r + \frac{x^2}{r^2}u_{rr} - \frac{xy}{r^3}u_{r\theta} + \frac{2xy}{r^4}u_\theta - \frac{xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta}$$

$$u_{xx} = \frac{x^2}{r^2} u_{rr} - \frac{2xy}{r^3} u_{r\theta} + \frac{y^2}{r^4} u_{\theta\theta} + \frac{y^2}{r^3} u_r + \frac{2xy}{r^4} u_\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

$$u_{yy} = \frac{y^2}{r^2} u_{rr} + \frac{2xy}{r^3} u_{r\theta} + \frac{x^2}{r^4} u_{\theta\theta} + \frac{x^2}{r^3} u_r - \frac{2xy}{r^4} u_\theta \qquad \ldots\ldots\ \ldots\ldots\ldots\ldots\ldots(10)$$

Adding equations (9) and (10) we get

$$u_{xx} + u_{yy} = \frac{x^2+y^2}{r^2} u_{rr} + \frac{x^2+y^2}{r^4} u_{\theta\theta} + \frac{x^2+y^2}{r^3} u_r$$

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r$$

$$\textit{Laplace equation in polar form is } u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0$$

PROBLEMS

1. Show that the only solutions of Laplace equation depending only on r is u = a ln r + b

   Solution:

   $$\textit{Laplace equation in polar form is } u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0$$

   As u depends only on r, u is a function of r only.

   Hence $u_\theta=0$ and $u_{\theta\theta} = 0$

   $$\textit{Hence } u_{rr} + \frac{1}{r} u_r = 0 \quad \textit{or} \quad u_{rr} = -\frac{1}{r} u_r$$

   $$\textit{Let } u_r = p \textit{ then } u_{rr} = \partial p / \partial r$$

   $$\textit{Hence } \frac{\partial p}{\partial r} = -\frac{p}{r} \quad \textit{or} \quad \frac{\partial p}{p} = -\frac{\partial r}{r}$$

   Integrating both sides we get ln p = - ln r + ln a

   $$\textit{Hence } p = \frac{\partial u}{\partial r} = \frac{a}{r} \quad \textit{or} \quad \partial u = \frac{a \partial r}{r}$$

   Integrating again both sides we get ln u = a ln r + b

2. Find the electrostatic potential (Steady state temperature distribution) in the disk r<1 corresponding to the boundary values $4\cos^2\theta$

Solution

The boundary value $f(\theta) = 4\cos^2\theta$ which is even function, $-\pi < \theta < \pi$ .Hence Bn = 0

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\mathrm{Cos}^2\theta d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} [1 + \mathrm{Cos}2\theta] d\theta = \frac{1}{\pi}[\theta + 0.5\,\mathrm{Sin}\,2\theta]_{-\pi}^{\pi} = 2$$

$$An=\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{f}(\theta)\,\mathrm{Cos\,n}\theta d\theta=\frac{1}{\pi}\int_{-\pi}^{\pi}4\mathrm{Cos}^2\theta\,\mathrm{Cos\,n}\theta\ d\theta=\frac{1}{\pi}\int_{-\pi}^{\pi}2[1+\mathrm{Cos}2\theta]\mathrm{Cos\,n}\theta\ d\theta$$

$$=\frac{2}{\pi}\int_{-\pi}^{\pi}\mathrm{Cos\,n}\theta\ d\theta+\frac{2}{\pi}\int_{-\pi}^{\pi}\mathrm{Cos\,n}\theta\ \mathrm{Cos}2\theta\ d\theta$$

$$=\frac{2}{n\pi}[\mathrm{Sin\ n}\theta]_{-\pi}^{\pi}+\frac{1}{\pi}\int_{-\pi}^{\pi}[\mathrm{Cos}(\mathrm{n}\theta-2\theta)+\mathrm{Cos}(\mathrm{n}\theta+2\theta)]d\theta$$

$$=0+\frac{1}{\pi}\left[\frac{Sin(\mathrm{n}\theta-2\theta)}{n-2}+\frac{Sin(\mathrm{n}\theta+2\theta)}{n+2}\right]_{-\pi}^{\pi}\ =0\ (except\ n=2)$$

$$For\ n=2,\ \ An=A2=\frac{1}{\pi}\int_{-\pi}^{\pi}2[1+\mathrm{Cos}2\theta]\mathrm{Cos}\,2\theta\ d\theta$$

$$=\frac{2}{\pi}\int_{-\pi}^{\pi}\mathrm{Cos}\,2\theta\ d\theta+\frac{2}{\pi}\int_{-\pi}^{\pi}\mathrm{Cos}^2\ 2\theta\ \ d\theta=\frac{2}{2\pi}[\mathrm{Sin}\ 2\theta]_{-\pi}^{\pi}+\frac{1}{\pi}\int_{-\pi}^{\pi}[1+\mathrm{Cos}\,4\theta]d\theta$$

$$=0+\frac{1}{\pi}\left[\theta+\frac{Sin4\theta}{4}\right]_{-\pi}^{\pi}\ =2$$

The electrostatic potential (Steady state temperature distribution) in the disk

$$u\ (r,\theta)\ =A_0+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n[A_nCos\,n\theta+B_nSin\ n\theta]=2+2r^2Cos2\theta$$

(Since R= radius of disk=1 and B_n= 0)

11.10 **CIRCULAR MEMBRANE**

Two dimensional wave equation is $u_{tt} = c^2\ (u_{xx} + u_{yy}) = c^2\ \nabla^2 u$

Using Laplacian in polar form we have

$$Laplace\ equation\ in\ polar\ form\ is\ \nabla^2u=u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r=0$$

$$\frac{\partial^2u}{\partial t^2}=c^2\left[u_{rr}+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r}u_r\right]$$

AS circular membrane is radially symmetric, u depends on r only and u does not depend on θ

Hence u_θ=0 and $u_{\theta\theta}$ = 0

$$Hence\ \frac{\partial^2u}{\partial t^2}=c^2\left[u_{rr}+\frac{1}{r}u_r\right] \qquad \text{..................................(1)}$$

Boundary Condition u(R, t) = 0 ....(2)

Initial Conditions: $u(r,0) = f(r)$ = initial deflection ....(3)

$u_t(r,0) = g(r)$ = initial velocity ....(4)

<u>Step I</u> Let $u(r,t) = F(r)G(t)$

$u_r= F'(r)\, G(t)$, $u_{rr}= F''(r)G(t)$ and utt= FG**

where ' and * represents partial differentiation w.r.to r and t respectively.

Putting in equation (1) we get

$F(r)G^{**}(t)= c^2[F''(r)G(t)+(1/r)\, F'(r)\, G(t)]$

$G^{**}(t)/[\, c^2G(t)+(]= [F''(r) + (1/r)\, F'(r)]/\, F(r)$ ....(5)

L.H.S. involves function of t only and R.H.S. involves function of r only. Hence both expressions must be equal to some constant D.

For $D \geq 0$, as G= 0, hence u = o and we can not get solution.

For $D < 0$ let $D = - k^2$ (negative)

$G^{**}(t)/\, [c^2\, G(t)] = [F''(r) + (1/r)\, F'(r)]/\, F(r)= - k^2$ = constant

$G^{**} + k^2c^2G = 0$

or $G^{**} + \lambda^2G = 0$ where $\lambda= c\, k$ ....(6)

and $F'' + (1/r)\, F' + k^2\, F = 0$ ....(7)

Put $s = kr$ then $1/r = k/s$ implies $ds/dr=k$

$F' =dF/dr=dF/ds.\ ds/dr = k\, dF/ds$

$$F'' = \frac{\partial^2 F}{\partial r^2} = \frac{\partial}{\partial r}\left(k\frac{\partial F}{\partial s}\right) = k\frac{\partial}{\partial s}\left(\frac{\partial F}{\partial s}\right)\frac{\partial s}{\partial r} = k^2\frac{\partial^2 F}{\partial s^2}$$

Equation (7) becomes

$$k^2\frac{\partial^2 F}{\partial s^2} + \frac{k^2}{s}\frac{\partial F}{\partial s} + k^2 F = 0$$

$$\frac{\partial^2 F}{\partial s^2} + \frac{1}{s}\frac{\partial F}{\partial s} + F = 0$$

This is **Bessel's equation.**

Solution is $F(r) = J_0(s) = J_0(kr)$ ....(8)

On the boundary $r=R$ hence $F(r) = J_0(kr) = 0$

$J_0(s)$ has infinitely many positive roots,

$s= \alpha_1,\alpha_2,\alpha_3,$..................

α_1=2.404,α_2=5.52,α_3=8.653,..................

From (8) kR= α_m and k= α_m r/R ....(9)

Fm(r)= $J_0(k_m r)$= $J_0(\alpha_m r/R)$ ....(10)

General solution of (6) is $G_m(t) = a_m \cos \lambda_m t + b_m \sin \lambda_m t$

Hence $u_m(r,t)= F_m(r)\, G_m(t)=(a_m \cos \lambda_m t + b_m \sin \lambda_m t)\, J_0(k_m r)$, m=1,2,3..... ....(11)

are solution of wave equation (1). These are eigen functions. The corresponding eigen values are

$\lambda_m = c\, \alpha_m /R$

The vibration of membrane corresponding to um is called m[th] normal mode.

Step III

We consider the series

$$u\ (r,t) = \sum_{m=1}^{\infty} F_m(r) G_m(t) = \sum_{m=1}^{\infty} \left[a_m Cos\lambda_m t + b_m\ Sin\lambda_m t\right] J_0\left(\frac{\alpha_m r}{R}\right) \text{.................(12)}$$

Putting t=0 we get $u(r,0) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m r}{R}\right) = f(r)$ = $J_0(\alpha_m r/R)$ ....(13)

The series (12) will satisfy initial condition (3) provided the constant am must be coefficient of the Fourier –Bessel series(13).

$$a_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r\ f(r) J_0(\alpha_m r / R) dr$$

$$Deflection\ u(r,t)\ = \sum_{m=1}^{\infty}\left[a_m\ Cos\lambda_m t + b_m Sin\lambda_m t\right] J_0\left(\frac{\alpha_m r}{R}\right)$$

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty}\left[-\lambda_m a_m\ Sin\lambda_m t + b_m \lambda_m Cos\lambda_m t\right] J_0\left(\frac{\alpha_m r}{R}\right)$$

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = \sum_{m=1}^{\infty}\left[b_m \lambda_m\right] J_0\left(\frac{\alpha_m r}{R}\right) = g(r)$$

$$b_m = \frac{2}{c\alpha_m R J_1^2(\alpha_m)} \int_0^R r g(r) J_0(\alpha_m r / R) dr$$

PROBLEM

1. Find the deflection of the drum with R=1, c=1 if the initial velocity is 1 and initial deflection is 0

Solution:

Given R=1, c=1 and g(r)=1 .

Given initial deflection =f(r)= 0 , hence we have a_m=0

$\lambda_m = c\alpha_m / R = \alpha_m$ (as given R=1, c=1)

$\lambda_1 = \alpha_1 = 2.404$, $\lambda_2 = \alpha_2 = 5.52$, $\lambda_3 = \alpha_3 = 8.653$

The initial velocity =g(r)=1

$$b_m = \frac{2}{c\alpha_m R J_1^2(\alpha_m)} \int_0^R r g(r) J_0(\alpha_m r / R) dr$$

$$= \frac{2}{\alpha_m J_1^2(\alpha_m)} \int_0^1 r J_0(\alpha_m r) dr$$

From properties of Bessel's function

$$x^n J_{n-1}(x)\ dx = x^n J_n(x)$$
$$Hence\ \ r^n J_{n-1}(r)\ dr = r^n J_n(r)$$
$$Putting\ n = 1\ \ we\ get\ \ rJ_0(r)\ dr = rJ_1(r)$$

$$b_m = \frac{2}{\alpha_m J_1^2(\alpha_m)} \left[\frac{rJ_1(\alpha_m r)}{\alpha_m}\right]_0^1 = \frac{2}{\alpha_m^2 J_1^2(\alpha_m)} [J_1(\alpha_m)] = \frac{2}{\alpha_m^2 J_1(\alpha_m)}$$

$$Deflection\ u(r,t) = \sum_{m=1}^{\infty} [a_m\ Cos\lambda_m t + b_m Sin\lambda_m t] J_0\left(\frac{\alpha_m r}{R}\right)$$

$$= \sum_{m=1}^{\infty} [b_m Sin\lambda_{mn} t] J_0\left(\frac{\alpha_m r}{R}\right) = \sum_{m=1}^{\infty} \left[\frac{2}{\alpha_m^2 J_1(\alpha_m)} Sin\lambda_m t\right] J_0(\alpha_m r)$$

11.11 Laplace equation in Cylindrical and spherical coordinate

Cylindrical coordinate

x= r Cos θ, y= r Sin θ, z=z

$$Laplace\ \ equation\ \ in\ cylindrica\,l\ \ form\ \ is\ \nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = 0$$

Spherical coordinate

x= r Cos θ Sin ϕ, y= r Sin θ Sin ϕ, z=r Cos ϕ

$$Laplace equation in\ spherical form\ \ is\ \nabla^2 u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} u_{\phi\phi} + \frac{Cot\phi}{r^2} u_\phi + \frac{1}{r^2 Sin^2\phi} u_{\theta\theta} = 0$$

Potential in the interior of sphere

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(Cos\phi)$$

$$where\ A_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\phi)Pn(Cos\phi)Sin\phi\ d\phi$$

Potential in the Exterior of sphere

$$u(r,\phi) = \sum_{n=0}^{\infty} B_n r^{-n-1} - P_n(Cos\phi)$$

$$where\ B_n = \frac{2n+1}{2}R^{n+1}\int_0^{\pi} f(\phi)Pn(Cos\phi)Sin\phi\ d\phi$$

PROBLEMS

1. Show that the only solutions of Laplace equation depending only on r is u = c/r + k where $r^2= x^2 +y^2 +z^2$

   Solution:

   $$Laplace equationin\ spherical form\ is\ \nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + \frac{Cot\phi}{r^2}u_\phi + \frac{1}{r^2 Sin^2\phi}u_{\theta\theta} = 0$$

   As u depends only on r, u is a function of r only.

   Hence $u_\theta=0$ and $u_{\theta\theta} = 0$, $u_{\phi\phi} = 0$

   $$Hence\ u_{rr} + \frac{2}{r}u_r = 0 \quad or \quad u_{rr} = -\frac{2}{r}u_r$$

   $$Let\ u_r = p\ then\ u_{rr} = \partial p/\partial r$$

   $$Hence\ \frac{\partial p}{\partial r} = -\frac{2p}{r} \quad or \quad \frac{\partial p}{p} = -\frac{2\partial r}{r}$$

   Integrating both sides we get ln p = - 2ln r + ln c

   $$Hence\ p = \frac{\partial u}{\partial r} = \frac{c}{r^2} \quad or \quad \partial u = \frac{c\partial r}{r^2}$$

   Integrating again both sides we get ln u = $D(-r^{-1}) + k$

   Or u= c/r + k where c= -D

2. Find the Potential in the interior of sphere , R=1 assuming no charges in interior of sphere and potential on surface is f(ϕ)= Cos ϕ

   Solution: Given R=1 and f(ϕ)= Cos ϕ

$$A_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\phi)Pn(Cos\phi)Sin\phi\ d\phi$$

$$A_n = \frac{2n+1}{2}\int_0^{\pi} Cos\ \phi\, Pn(Cos\phi)Sin\phi\ d\phi$$

Putting Cos ϕ= x, -Sin ϕ dϕ = dx

As ϕ→0, x→1 and as ϕ→π, x→ -1 Hence we have

$$A_n = \frac{2n+1}{2}\int_{1}^{-1} - x\,P_n(x)\,dx \;=\frac{2n+1}{2}\int_{-1}^{1} x\,P_n(x)\,dx = \frac{2n+1}{2}\int_{-1}^{1} P_1(x)P_n(x)\,dx$$

$$= \frac{2n+1}{2}\begin{bmatrix} \dfrac{2}{2n+1}\; for\; n=1 \\ 0\;\; for\; n \neq 1 \end{bmatrix} \quad == \begin{bmatrix} 1 \quad for\; n=1 \\ 0\;\; for\; n \neq 1 \end{bmatrix}$$

Hence $A_1 = 1$, $A_2 = 0$, $A_3 = 0$, $A_4 = 0$

$$u(r,\phi) \;= \sum_{n=0}^{\infty} A_n r^n P_n(Cos\phi)$$
$$= A_0 P_0 + A_1 r P_1(Cos\phi) + A_1 r^2 P_2(Cos\phi) + \ldots\ldots\ldots$$
$$= rP_1(Cos\phi) = rCos\phi$$

SOLUTION OF PDE BY LAPLACE TRANSFORM

Procedure:

Step I: We take the Laplace transform w.r. to one of the two variables usually t which gives an ODE for transform of the unknown function. It includes given boundary and initial conditions.

Step II: Solve the ODE and get the transform of the unknown function.

Step II: Taking the inverse Laplace transform the solution of the given problem will be obtained.

PROBLEM

1. Solve the PDE using Laplace transform

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = xt,\; u(x,0) = 0 \;and\;\; u(0,t) = 0 \;if\;\; t \geq 0$$

Solution

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = xt \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Taking the Laplace transform of both sides of equation (1) we get

$$xL\left(\frac{\partial u}{\partial x}\right) + L\left(\frac{\partial u}{\partial t}\right) = xL(t)$$

$$x\int_0^\infty \frac{\partial u}{\partial x}e^{-st}\,dt + sL(u) - u(x,0) = \frac{x}{s^2}$$

$Use\ \ formula\ L(dy/dt) = sL(y) - y(0)\ \ for\ derivative\ of\ \ Laplace\ transform$

Using definition of Laplace transform we have $L\left(\frac{\partial u}{\partial x}\right) = \int_0^\infty \frac{\partial u}{\partial x}e^{-st}\,dt$

Assuming that we may interchange differentiation and integration we have

$$x\frac{\partial}{\partial x}\int_0^\infty ue^{-st}\,dt + sL(u) - 0 = \frac{x}{s^2}\ \ Since\ given\ u(x,0) = 0$$

$$x\frac{\partial}{\partial x}L(u) + sL(u) = \frac{x}{s^2}$$

$$x\frac{\partial U}{\partial x} + sU = \frac{x}{s^2} \qquad where\ \ U = L(u)$$

$$\frac{\partial U}{\partial x} + \frac{s}{x}U = \frac{1}{s^2}$$

Which is a first order linear differential equation with p=s/x and q= $1/s^2$

Integrating factor F= $e^{\int p dx}=e^{\int (s/x) dx}= e^{s\ln x}= e^{\ln x^s}= x^s$

The solution is U=(1/F)[∫F.Q dx +c] = x^{-s}[∫x^s/s^2 dx +c] = x^{-s}[∫$x^{s+1}/(s^2 (s+1))$ +c]

$$U = \frac{x}{s^2(s+1)} = L(u)\ \ Hence\ u = L^{-1}\left(\frac{x}{s^2(s+1)}\right) = xL^{-1}\left(\frac{1}{s^2(s+1)}\right) = xL^{-1}\left(\frac{s^2-(s^2-1)}{s^2(s+1)}\right)$$

$$= xL^{-1}\left(\frac{1}{(s+1)} - \frac{s-1}{s^2}\right) = xL^{-1}\left(\frac{1}{(s+1)} - \frac{1}{s} + \frac{1}{s^2}\right) = x\left(e^{-t} - 1 + t\right)$$

UNIT-III & UNIT-IV

1 Origin of complex number and complex analysis

Euler in 1748 derived the formula $e^{i\theta} = \cos\theta + \sin\theta$ $e^{i\pi} = -1$, A fantastic relation that include the three symbols e, π, i in one surprising equation. Complex number is a point in the plane. This idea is attributed to Argand who wrote it up indipendently in 1806. Due to this geometric interpretation of complex numbers is known on Argand diagram.

Just as solutions of real quadratic equations could lead to new complex numbers, so the solutions of equations with complex coefficients lead to even more kinds of new numbers.

Jean D'Alembert $(1717 - 1783)$ conjecture that complex numbers alone would suffice.

Gauss confirmed this in the Fundamental theorem of Algebra-"every polynomial equation has a complex root".

In 1837, nearly three centuries after Cardan's use as imaginary numbers, willam Roman Hamilton published definition of complex numbers as ordered pair of real numbers subject to certain explicit rules of manipulation.

Gauss wrote to woltgang that he had developed the same idea 1831.

For centuries it is believed that complex analysis is an incredible complicated theory.

It took almost three centuries to obtain satisfactory treatment of complex number. It then took less than tenth of that line to complete a major part of complex analysis.

Once a breakthrough occurred, further development is easy. Complex numbers$\rightarrow$ complex analysis.

In 1545, Cardans solve the problem

$$x + y = 10$$

$$xy = 40$$

Here the solution is:

$$x = 5 + \sqrt{-15}$$

$$y = 5 - \sqrt{-15}$$

Cardans gave no interpretation for the square root of a negative number.
Solving

$$x^3 = 15x + 4$$

by Tartaglith formula tends to

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

in contrast to the obvious answer

$$x = 4$$

Rapheal Bombelli (1526-73) suggested a way to reconcile the two solution:

$$(2 \pm \sqrt{-1})^3 = 2 \pm \sqrt{-121}$$

this makes Cardans expression

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4$$

this impossible root is a familiar root in a complex disguise.
La Geometric (1637) by Rene Descarte made distinction between real and imaginary numbers , representing imaginary numbers by a sign.
WAR BETWEEN LEIBNITZ AND BERNOULLI:
Leibnitz asserted that the logarithm of a negative number was complex whilst Bernoulli insisted it was real.Bernoulli argued since

$$\frac{d(-x)}{-x} = \frac{dx}{x}$$

It follows by integration that

$$\log(-x) = \log x.$$

Leibnitz insisted that this is true only for positive x. Euler resolve the controversy favor of Leibnitz in 1749 pointing out that the integration required arbitrary constant, a point Bernoulli has ignored.

i is a imaginary number , which does not lie in $\mathbf{R}$ such that$i^2 = -1$.In other words, $i = \sqrt{-1}$. Based on this we from a new number

$$x + iy, x, y \in \mathbf{R}.$$

We call such a number as a complex number. Moreover , x is called as real part and y is called as imaginary part of the complex number,i.e

$$x = Re(x + iy), y = Im(x + iy)$$

The set of all complex numbers is denoted by the symbol$\mathbf{C}$,

$$\mathbf{C} = \{x + iy : x, y \in \mathbf{R}, i = \sqrt{-1}\}.$$

We denote a particular complex numbers is denoted by the symbol,

$$z = x + iy$$

This set is an extension of $\mathbf{R}$.The set of real numbers ,as every real number is a number of $\mathbf{C}$. Moreover the complex numbers obeys many of the some rules of arithmetic numbers . We list them as follows:
addition : (a+ib)+(c+id)=(a+c)+i(b+d)
multiplication : (a+ib).(c+id)= (ac-bd)+i(ad+bc)
other properties are

$$z + w = w + z$$

$$z.w = w.z$$

$$z + (u + v) = (z + u) + v$$

$$(zw).u = z.(wu)$$

$$z.(w + u) = zw + zu$$

$$z + 0 = 0 + z = z$$

$$z.1 = 1.z = z$$

GEOMETRY:
Our way to represent a comlex numbers a+ib is by a point (a,b) in the plane $\mathbf{R}^2$. a+ib

can also be represented by the vector $a\hat{\imath} + b\hat{\jmath}$ where $\hat{\imath} = (1, o)$ and $\hat{\jmath} = (0, 1)$.
clearly, a+ib is the vector whose initial point is (0,0) the origin and terminal point is (a,b). with this vector form representation **C** is vector space. One of the important concept in analysis is the concept of distance, equivalently called magnitude or norm of a vector.

MAGNITUDE:

The magnitude of a+ib is denoted by —a+ib—, and is defined by

$$|a + ib| = \sqrt{a^2 + b^2}$$

If z= a+ib, then $|z| = \sqrt{a^2 + b^2}$, which is also the distance of the point $(a + ib) \in \mathbf{R}^2$ from the origin. Moreover, $|a + ib|$ is also the length of the vector (a,b). The notation $|z|$ is called as the modulus of z.

CONJUGATE:

The complex conjugate (or just conjugate) of a+ib is the number $\overline{a + ib}$ defined by

$$\overline{a + ib} = a - ib$$

DIVISION:

Let a+ib and c+id are two complex numbers then,

$$\frac{a + ib}{c + id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

Ex:

Division of $2 - 7i$ by $8 + 3i$ is the complex number

$$\frac{2 - 7i}{8 + 3i} = -\frac{5}{73} - i\frac{62}{73}$$

POLAR FORM:

Let $z = a + ib$, the number z is anonymous to the cartesian coordinate (a,b). Which has polar coordinates (r, θ). We have, $r = |z|$ and $\theta =$ argument of z. so

$$a = r\cos\theta, b = r\sin\theta$$

Euler formula says

$$e^{i\theta} = \cos\theta + \sin\theta$$

$$\Rightarrow z = re^{i\theta}$$

Ex:
Polar form of $(-1+4i)$ is,

$$-1+4i = \sqrt{17}e^{i(\pi - \tan^{-1}(4))}$$

.

2 SOME THEOREMS , DEFINITIONS , FORMS

De Movire's Theorem: For any integer n.
(a) $(\cos\theta + i\sin\theta)^n = \cos n\theta$.
(b) If $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.
A complex number z is given, we can now define polynomials of degree n (say),

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n, a_n \neq 0.$$

FUNDAMENTAL THEOREM OF ALGEBRA:
A polynomial of degree n with complex coefficient has at most n complex roots.For example the polynomial $z^2 - 1$ has two roots , in fact they are solutions of the equation $z^2 - 1 = 0$. It is not difficult to find that they are $1, -1$. If we consider $z^3 - 1$ then by fundamental theorem of algebra it has at most three complex roots.
We are basically concentrating on degree 2. So try to solve these problems.
(a) $az^2 + bz + c = 0$.
(b) $az^4 + bz^2 + c = 0$.
(c) $z^2 + z + 1 - i = 0$.
(d) $z^4 - (1+4i)z^2 + 4i = 0$.
LOCUS:
Now we define the locus of some standard curves,
(a) $|z| = 1$ represents the locus of unit circle.
(b) $|z| \leq 1$ represents the locus of a closed unit disk.
(c) $|z| < 1$ represents the locus of open unit disk.

(d) $\frac{1}{2} \leq |z| \leq 1$ represents the locus of annulus.

General equation of a circle:

$$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0, \beta is a real number$$

$$|z + \alpha|^2 = \alpha\bar{\alpha} - \beta$$

it represents a circle provided $\alpha\bar{\alpha} - \beta > 0$.

General equation of a straight line:

$$\bar{\alpha}z + \alpha\bar{z} + \beta = 0, \alpha \neq 0, \beta is real.$$

CIRCLES:

Consider the equation $|z-a| = r$ then locus of points satisfying this equation is the circle of radius r about a.

OPEN DISK, NEIGHBOURHOOD:

The inequality $|z - a| < r$ specifies all points within the disk of radius r and center a. It is also called a neighbourhood of a.

CLOSED DISK:

$|z - a| \leq r$, consists of all points on or within the circle of radius about r.

STRAIGHT LINE:

$|z - a| = |z - b|$

Perpendicular bisector of the line segment joining a and b.

EX:

Find cartesian form of the straight line defined by the equation

$$|z + 6i| = |z - 1 + 3i|$$

Ans:

$$|z + 6i|^2 = |z - 1 + 3i|^2$$

$\Rightarrow$

$$z\bar{z} + 6i(z - \bar{z}) + 36 = z\bar{z} - (z + \bar{z}) - 3i(z - \bar{z}) + 1 + 3i - 3i + 9$$

$\Rightarrow 12y = -2x + 6y - 26$

$\Rightarrow$

$$y = -\frac{1}{3}(x + 13)$$

INTERIOR POINTS, BOUNDARY POINTS, OPEN AND CLOSED SETS:

• A complex number z_0 is an interior point of a set S if there is a neighbourhood of z_0 containing only points of S.

• S is a open if every point of S has a neighbourhood containing only points of S.

• A point z_0 is a boundary point of S if every neighbourhood of z_0 contains at least one point in S and at least one point not in S.

• S is an open set if every point of S is an interior point.

• S is a closed set if its complement S^c is open.

• S is closed if it contains all of its limit points.

• S is closed if and only if S contains all its limit points.

DEFINITION:

A point a is called a limit point of S(may or may not belongs to S) if every neighbourhood of a contains at least one point of S differing from a.

Let $S \subseteq C$ and $z_0 \in C$. Then z_0 is called a limit point of S if every nbd of z_0 contains infinitely many points of S.

Sequence:

A complex sequence $\{z_n\}$ is an assignment of a complex number z_n to each positive integer n.

Convergence:

A complex sequence $\{z_n\}$ converges to the number L if, given any positive number ϵ , there is a positive number N such that,

$$|z_n - L| < \epsilon, if, n \geq N.$$

EX:

• The sequence $\{1 + \frac{i}{n}\}$ converges to 1.

• The sequence $\{(-1)^n + \frac{i}{n}\}$ has two limit points 1 and -1 , and they are not equal . Hence , the sequence does not converge.

THEOREM:

Let $z_n = x_n + iy_n$ and $L = a + ib$, Then $z_n \to L$ if and only if $x_n \to a$ and $y_n \to b$.

Subsequence:

A subsequence of a sequence is formed by picking out certain terms to form a new sequence.

Bounded Sequence:

A complex sequence $\{z_n\}$ is bounded if $|z_n| \leq M, \forall n = 1, 2, ...$

Theorem:

The sequence $\{z_n\}$ is bounded if the sequence $\{z_n\}$ has a convergent subsequence.

Compact Set:

A set K of complex number is compact if it is closed and bounded.

Bolzano-Weirstrass Theorem:

Let K be an infinite compact set of complex numbers. Then K contains a limit point.

Series:

Power Series: A Power Series in powers of $z - z_0$ is a series of the form $\displaystyle\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + ...$, $a_0, a_1, ...$ are called the co-efficient series and z_0 is the center of the series.

Convergence of Power series: (i) Every power series $\displaystyle\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges at the center z_0.

(ii) If the above power series converges at a point $z = z_1 \neq z_0$, it converges absolutely for every z closer to z_0 than z_1, i.e. $|z - z_0| < |z_1 - z_0|$.

Radius of convergence of power series: Consider the smallest circle center z_0 that includes all the points at which a given power series converges. Let R denote its radius. Thecircle $|z - z_0| = R$ is called the circle of convergence and its radius R, the radius of convergence of the given power series.

Remark: Termwise differentiation and integration of the power series is permissible.

Taylor Series: The Taylor series of a function $f(z)$, the complex analog of the real Taylor series is $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n(z - z_0)^n$, where $a_n = \frac{1}{n!} f^n(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$, C : simple closed path that contains z_0, counter clockwise sense.

The remainder term of the above series after the term $a_n(z - z_0)^n$ is

$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$.
Therefore $f(z) = f(z_0) + \frac{(z-z_0)}{1!} f' z_0 + \frac{(z-z_0)^2}{2!} f'' z_0 + ... + \frac{(z-z_0)^n n}{n!} f^n z_0 + R_n(z)$ is called Taylor's formula with remainder term.
Remark: A Maclaurin series is a Taylor series with center $z_0 = 0$.
Laurent Series: Let $z = z_0$ is an isolated singularity of f. Then $f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n$ be its Laurent series expansion in $a_{nn}(a, r, R)$.
Now $f(z) = \sum_0^{\infty} a_n(z-z_0)^n + \sum_1^{\infty} b_n(z-z_0)^{-n}$,
where $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-z_0)^{n+1}} dw$, $b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-z_0)^{1-n}} dw$.

3 Limit, Continuity, Derivative

Function:
A function f is defined on S is a rule that assigns to every z in S a complex number w, we can write it as,

$$w = f(z)$$

or,

$$w = f(z) = u(x, y) + iv(x, y)$$

where u(x,y) and v(x,y) are functions of variables x and y.
Limit:
Let $f : S \to \mathbf{C}$ be a complex function , let z_0 be a limit point of S and L be a complex number. Then

$$\lim_{z \to z_0} f(z) = L$$

if and only if given $\epsilon > 0$, there exist a positive number δ such that $|f(z) - L| < \epsilon$ for all z in S such that $0 < |z - z_0| < \delta$.
Continuity:
•(Limit form) Let f be a complex valued function defined on a region D of the complex plane . Let $z_0 \in D$ then f is said to be continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

• $(\epsilon - \delta form)$ Let f be a complex valued function defined on a region D of the complex plane . Let $z_0 \in D$, then f is said to be continuous at z_0 if given $\epsilon > 0$ there exist $\delta > 0$

such that,

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon.$$

• f is said to be continuous in D if it is continuous at each point of D.

• If a function f is continuous at all z for which it is defined , then f is a continuous function.

Theorem:

The image of a closed and bounded set under a continuous function is also closed and bounded.

EX:

• The function $f(x) = \frac{1}{x}$ in $(0, 1)$ is unbounded.

• The function $f(x) = |x|$ is unbounded on $\mathbf{R}$.

• The function $f(x) = \{^{1,x\in\mathbf{Q}}_{0,x\in\mathbf{R\text{-}Q}}.$ is continuous.

Derivative:

The derivative of a complex function $f : D \to C$ at a point $z_0 \in D$ is written as $f'(z_0)$ and is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided that the limit exists. Then f is said to be differentiable at z_0.

EX:

• $f(z) = z^2$ is differentiable for all z.

• $\bar{z}$ is nowhere differentiable.

Theorem:

If $f(z)$ is differentiable at z_0 then it is continuous at that point.

Corollary: Converse is not true; counter example is $f(z) = \bar{z}$.

Ex: Find the derivative of the following function $f(z) = \frac{z-1}{z+1}$

Ans: Try this one.

Ex: Prove that $f(z) = Rez$ is nowhere differentiable.

Ans:

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{(\Delta x \to 0)(\Delta y \to 0)} \frac{x + \Delta x - x}{\Delta x + i\Delta y} \\ &= \lim_{(\Delta x \to 0)(\Delta y \to 0)} \frac{\Delta x}{\Delta x + i\Delta y} \end{aligned}$$

has no limit.

4 Analytic functions and some standard functions

Analytic Function:

(Analytic in a domain D) A function $f(z) : D \to C$ is said to be analytic in a domain D if f(z) is defined and differentiable at all points of D.

(Analytic at a point) A function $f(z) : D \to C$ is said be analytic at a point $z = z_0$ in D if f(z) is analytic in a neighbourhood of z_0.

Ex:

• $(z^3 + z)$ is analytic.(entire function).

• Examples of not analytic functions (1) $f(z) = Rez$. (2) $f(z) = Imz$. (3) $f(z) = |z|^2$.

Remark: If f is analytic at a point z_0. But converse is not true.

Ex: (1) $f(z) = \bar{z}$ is nowhere differentiable so not analytic.

(2) $f(z) = |z|^2$ is not an analytic function.

Remark: Set of all points for which a given function is analytic forms an open set.

Cauchy Riemann Equations:

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighbourhood of a point $z = x + iy$ and differentiable at z itself. Then at that point the first order partial derivatives u_x,u_y,v_x,v_y,exists and satisfy cauchy riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Ex: 1 Test the functions for analyticity.

z^3, $e^x(\cos y + i \sin y)$, $e^{-x}(\cos y - i \sin y)$, $\frac{i}{z^5}$.

Ex: These following functions are not analytic,

(a) $f(z) = z|z|$, (b) $f(z) = i|z|^3$, (C) $f(x, y) = 2xy + i(x^2 + y^2)$.

Polar form of Cauchy Riemann equation:

If $f(z) = u(r, \theta) + iv(r, \theta)$be analytic at $z = r \cos\theta + ir \sin\theta$ then the Cauchy-Riemann equation has polar form,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Ex: Prove that $f(z) = z^2$ is analytic.

Ans: Let $z = x + iy$, then $f(z) = (x + iy)^2 = x^2 + y^2 + i2xy$,

Here $u = x^2 - y^2$ and $v = 2xy$

$\frac{\partial u}{\partial x} = 2x, \frac{\partial v}{\partial y} = 2x$, $\frac{\partial u}{\partial y} = -2y$, $\frac{\partial v}{\partial x} = 2y$. hence we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Laplace equation:

If $f(z) = u + iv$ is analytic in domain D then Laplace equation are satisfied ,

$$i.e, u_{xx} + u_{yy} = 0$$

and

$$v_{xx} + v_{yy} = 0$$

Ex: Prove that $u = x^2 - y^2$ satisfies Laplace equation.

Ans: Here $u = x^2 - y^2$ then,

$$u_x = 2x, u_y = -2y, u_{xx} = 2, u_{yy} = -2$$

$$\Rightarrow u_{xx} + u_{yy} = 0$$

so u satisfies the laplace equations.

Harmonic function:

A function u(x,y) is called harmonic function if it satisfies Laplace equation.

Ex: $u = x^2 - y^2$ is a harmonic function but $v = x^2 + y^2$ is not a harmonic because $v_{xx} + v_{yy} \neq 0$

Conjugate harmonic function:

If $f(z) = u + iv$ is analytic then v is called the conjugate harmonic of u.

Note: If $f(z) = u + iv$ is analytic then u and v are harmonic.

Construction of analytic function if either u(x,y) or v(x,y)is given:

Using Thompson Milne method we can from the analytic function f(z) if either u or v

are given. If u is given and it is harmonic then its corresponding analytic function can be determined as follows.
step(i) find u_x and u_y
step(ii)

$$f'(z) = \frac{\partial u}{\partial x}]_{(z,0)} - i\frac{\partial u}{\partial y}]_{(z,0)}$$

step(iii) f(z) is obtained by integrating above $f'(z)$ in step (ii) w.r.to z.
Ex: If $u = x^2 - y^2$ is harmonic then find its corresponding analytic function.
Ans: $u = x^2 - y^2$, then $u_{xx} = 2$ and $u_{yy} = -2$
$\Rightarrow u_{xx} + u_{yy} = 0, \Rightarrow u$ satisfies Laplace equation. So u is harmonic.
now $u_x = 2x$, $u_y = -2y$

$$f'(z) = \frac{\partial u}{\partial x}]_{(z,0)} - i\frac{\partial u}{\partial y}]_{(z,0)}$$

$$\Rightarrow f'(z) = 2z - i(-2.0), \Rightarrow f'(z) = 2z$$

integrating we get,$f(z) = z^2 + c$
this is the required analytic function.
CONFORMAL MAPPING:
A conformal mapping is a mapping that preserves angles between any oriented curves both in magnitude and in sense.
THEOREM: The mapping defined by an analytic function f(x) is conformal ,except at critical points, that is points at which the derivative f'(z) is zero.
proof: Try this one.
Ex: e^x is conformal except at $z = 0$.
Ex: Consider the mapping $f(z) = \bar{z}$. It is not an analytic function but it represents reflection about the real axis and preserves the angle in magnitude but reverse the direction. Hence ,it is an isogonal mappings . Condition $f'(z_0) \neq 0$ can't be done. Since it is nowhere analytic.
Ex:

Find the angle made by the mapping $w = z^2$ at the point $z = 1 + i$.

Ans:

$w'(z) = 2z$, then required angle = arg $w'(z)]_{(z=1+i)} = arg = 2z]_{1+i} = \frac{\pi}{4}$.

Condition of conformality:

A mapping $w = f(z)$ is conformal at each point z_0 where f(z) is analytic and $f'(z_0) \neq 0$.

Linear Fractional Transformations:

Bilinear Transformation:

Bilinear Transformation is the function w of a complex variable z of the form

$$w = f(z) = \frac{az + b}{cz + d}$$

where a,b,c,d are complex or real constants subject to $ad - bc \neq 0$.

if $ad - bc = 0$, f(z) would be identically constant.

• For a choice of the constants a,b,c,d, we get special cases of bilinear transformation as

,

$$w = z + b \longrightarrow Translation.$$

$$w = az \longrightarrow Rotation.$$

$$w = az + b \longrightarrow Lineartransformation.$$

$$w = \frac{1}{z} \longrightarrow Inverseintheunitcircle.$$

Determination of Bilinear Transformation:

A bilinear transformation can be uniquely determined by three given conditions. Although four constants a,b,c,d appear in previously, essentially they are three ratios of these constants to the fourth one.

To find the unique bilinear transformation which maps three given distinct points z_1, z_2, z_3 onto three distinct images w_1, w_2, w_3. Hence the unique bilinear transformation that maps three given points z_1, z_2, z_3 onto three given images w_1, w_2, w_3, is given by

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

5 Complex Integration

Before going to discuss complex integral, we should aware about analytic function.

Analytic function: A complex variable function f is analytic in an open set if it has a derivative at each point in that set. If we say f is analytic in a set S which is not open, it is to be understood that f is analytic in an open set containing S. In particular, f is analytic at a point z_0 if it is analytic throughout some neighborhood of z_0.

• An **entire function** is a function that is analytic at each point in the entire finite plane. Every polynomial is an entire function.

• If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0, then z_0 is called a **singular point**, or **singularity**. For instance, the point $z = 0$ is a singular point of $f(z) = \frac{1}{z}$. The function $f(z) = |z|^2$, has no singular point because it is no where analytic.

Example 5.1 Every polynomial functions are analytic.

Example 5.2 The function $f(z) = \frac{1}{z}$ is analytic at each nonzero points in the finite plane.

Example 5.3 The function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$ and not throughout any neighborhood.

Integration in the complex plane is important for two reasons

• In many applications there occur real integrals that can be evaluated by complex integration, where as the usual methods of real integral calculus fail.

For example evaluations of the integrals

1. $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \frac{\Pi}{2\sqrt{2}}$
2. $\displaystyle\int_{-\infty}^\infty \frac{dx}{(1+x^2)^2} = \frac{\Pi}{2}$
3. $\displaystyle\int_{-\infty}^\infty \frac{\sin 3x}{1+x^4}dx = 0$
4. $\displaystyle\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\Pi}}{2}$

• Some basic properties of analytic functions can be established by complex integration, but would be difficult to prove by other method. The existence of higher derivatives of analytic functions is striking property of this type.

For example

1. Cauchy integral formula

2. Cauchy integral theorem etc

In the case of definite integral the path of integration is an interval on the real axis. In the case of complex definite integral, we shall integrate along a curve in the complex plane.

As in calculus we distinguish between definite and indefinite integrals or antiderivatives. An indefinite integral is function whose derivative equals a given analytic function in a region.

Complex definite integrals are called line integral and written as $\int_C f(z)dz$. Here the integrand $f(z)$ is integrated over a given curve C in the complex plane called the path of integration.

A curve C in the complex $z-$plane can be represented in the form

$$z(t) = x(t) + iy(t), \ t \text{ is a real parameter} \tag{5.1}$$

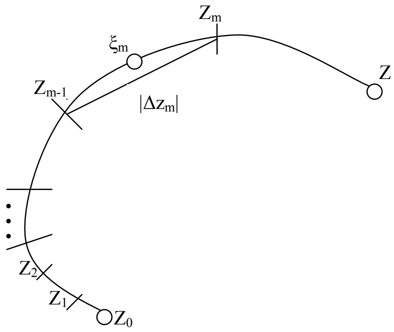
For example $z(t) = r\cos t + ir\sin t, \ |z| = r$.
The direction of increasing value of t in (5.1) is called the positive direction or positive sense on C. In this way (5.1) defines an orientation on C. We assume that $z(t)$ in (5.1) is differentiable and the derivatives $\dot{z}(t)$ is continuous with $\dot{z}(t) \neq 0$. The curve C has a unique tangent of its points called a smooth curve.

5.1 Definition of the complex line integral

This is similar to the method in calculus. Let C be a smooth curve in the complex plane given by (5.1)and f be continuous on C. We now subdivide (partition) the interval $a \leq t \leq b$ in (5.1) by points $t_0 = a, t_1, ..., t_{n-1}, t_n = b$, where $t_0 < t_1 < ... < t_n$. To this subdivision there corresponds a subdivision of C by points $z_0, z_1, ..., z_{n-1}, z_n$, where $z_j = z(t_j)$. On each portion of subdivision of C, we choose an arbitrary point, say ξ_1 between z_0 and z_1 (i.e. $\xi_1 = z(t), \ t_0 \leq t \leq t_1$), similarly ξ_2, ξ_3 etc. Then we form the sum

$$S_n = \sum_{m=1}^{n} f(\xi_m)\Delta z_m, \ \Delta z_m = z_m - z_{m-1}. \tag{5.2}$$

We do this for each $n = 2, 3, ...$ in a completely independent manner, but so that the greatest $|\Delta t_m| = |t_m - t_{m-1}|$ approaches zero as $n \to \infty$. This implies that the greatest $|\Delta z_m|$ also approaches zero because it can not exceed the length of the arc of C from z_{m-1} to z_m and the latter goes to zero since the arc length of the smooth curve C is continuous function of t. The limit of the complex numbers $S_2, S_3, ...$ thus obtained are called line integral of f over the oriented curve C. This curve C is called path of integration.



Complex line integral

The integral is denoted by

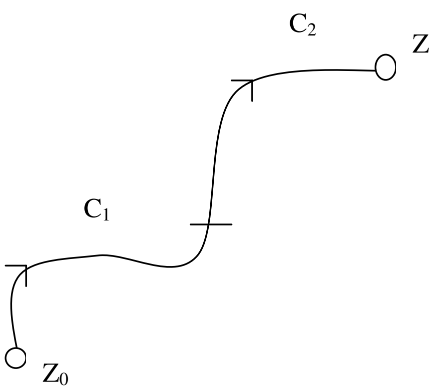
$$\int_C f(z)dz \text{ or } \oint_C f(z)dz, \text{ if } C \text{ is a closed path.} \tag{5.3}$$

- Basic properties

1. Linearity: $\displaystyle\int_C [k_1 f_1(z) + k_2 f_2(z)]dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz, \ k_1, k_2 \in \mathbb{C}.$

2. Sense reversal: $\displaystyle\int_{z_0}^{z} f(z)dz = -\int_{z}^{z_0} f(z)dz.$

3. Partition of path: $\displaystyle\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$



5.2 Existence of the complex line integral

Let f be continuous function and C be a piecewise smooth curve. Then the existence of the line integral $\int_C f(z)dz$ follows. Let $f(z) = u(x,y) + iv(x,y)$ set $\xi_m = \beta_m + i\eta_m,\ \Delta z_m = \Delta x_m + i\Delta y_m$.
Now (5.2) can be expressed as

$$S_n = \sum_{m=1}^{n} (u+iv)(\Delta x_m + i\Delta y_m),$$

where $u = u(\beta_m, \eta_m),\ v = v(\beta_m, \eta_m)$.
So

$$S_n = \sum_{m=1}^{n} u\Delta x_m - \sum_{m=1}^{n} v\Delta y_m + i\left[\sum_{m=1}^{n} u\Delta y_m + \sum_{m=1}^{n} v\Delta x_m\right]. \tag{5.4}$$

These sums are real. Since f is continuous, u and v are continuous. Hence if $n \to \infty$, then the greatest Δx_m and Δy_m approaches to zero and each sum on the right becomes a real line integral.

$$\lim_{n\to\infty} S_n = \int_C f(z)dz = \int_C udx - \int_C vdy + i\left[\int_C udy + \int_C vdx\right] \tag{5.5}$$

This shows that under our assumptions on f and C the line integral (5.3) exists and its value is independent of the choice of subdivisions and intermediate points ξ_m.

Theorem 5.1 Indefinite integral of analytic function
Let f be analytic in a simply connected domain D, i.e. there exists an indefinite integral of f such that $F'(z) = f(z)$ in D, and for all paths in joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0).$$

Example 5.4 $\int_0^i z^2 dz = \frac{1}{3}\left[z^3\right]_0^i = \frac{1}{3}i^3 = -\frac{1}{3}i$

Theorem 5.2 Integration by use of path
Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt,\ \dot{z} = \frac{dz}{dt}.$$

Steps applying in Theorem (5.2):

1. Represent the path C in the form $z(t)$, where $a \leq t \leq b$,
2. Calculate the derivative $\dot{z}(t) = \frac{dz}{dt}$,
3. Substitute $z(t)$ for every z in $f(z)$, and
4. Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

Example 5.5 Integrate $f(z) = \frac{1}{z}$ once arround the unit circle C in counter clockwise sense, starting from $z = 1$.
Solution: We may represent C in the form

$$z(t) = \cos t + i \sin t = e^{it}, \ 0 \leq t \leq 2\pi$$

$$\dot{z}(t) = -\sin t + i \cos t = ie^{it}.$$

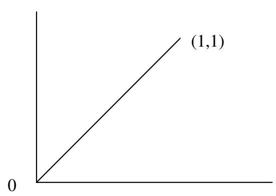
Now

$$\int_C \frac{1}{z} dz = \int_0^{2\pi} e^{-it}.i.e^{it} dt = \int_0^{2\pi} i dt = 2\pi i.$$

5.3 Bound for the absolute value integrals

Cauchy's inequality is given by $\left|\int_C f(z)dz\right| \leq ML$, where L is the length of C and M a constant such that $|f(z)| \leq M$ for $z \in C$.

Example 5.6 $\int_C z^2 dz$, where C is a straight line segment from 0 to $1+i$.
Now the length of $C = L = \sqrt{1+1} = \sqrt{2}$, $|f(z)| = |z^2| \leq 2$ on C.



By Cauchy's inequality,

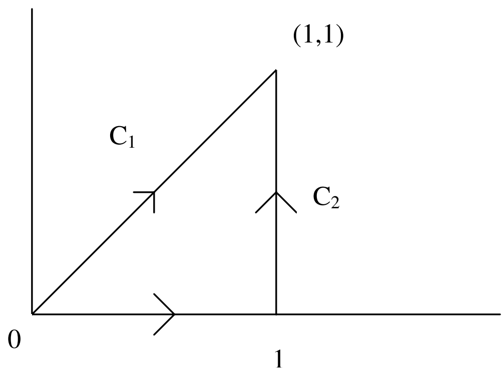
$$\left|\int_c z^2 dz\right| \leq 2\sqrt{2}.$$

Actual integration is $\int_C z^2 dz = \int_0^1 (t+it)^2(1+i)dt = \frac{2}{3}\sqrt{2}.$

6 Beauty of analytic functions on integration

If the function f is analytic on the domain D, then the integration is path independent i.e., the value of the integration gives same value for every smooth path C (end points of each path C are same) in D. That means the integration depend on end points only. But if the function is not analytic, then the value of integration is different for different path joining same initial and final points. We can see this things clearly from the following examples.

Example 6.1 Integrate $f(z) = z$ along the line segment from $z_0 = 0$ to $z = 1 + i$ for the paths C_1 and C_2.



The segment may be represent in the form

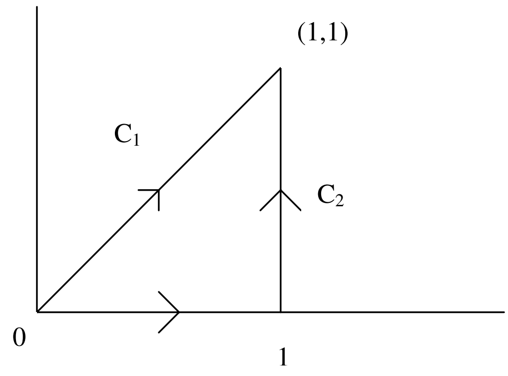
$$z(t) = x(t) + iy(t),\ 0 \leq t \leq 1$$

$$\int_{C_1} zdz = (1+i)\int_0^1 (1+i)tdt = \frac{1}{2}(1+i)^2 = i. \tag{6.6}$$

$$\int_{C_2} zdz = \int_0^1 tdt + \int_0^1 (1+it)idt = i. \tag{6.7}$$

From (6.6) and (6.7) we see that the value of the integration depends only on the end points as the function f is analytic.

Example 6.2 Integrate $f(z) = Re(z)$ along the line segment from $z_0 = 0$ to $z = 1 + i$ for the paths C_1 and C_2.



The segment may be represent in the form

$$z(t) = x(t) + iy(t),\ 0 \leq t \leq 1$$

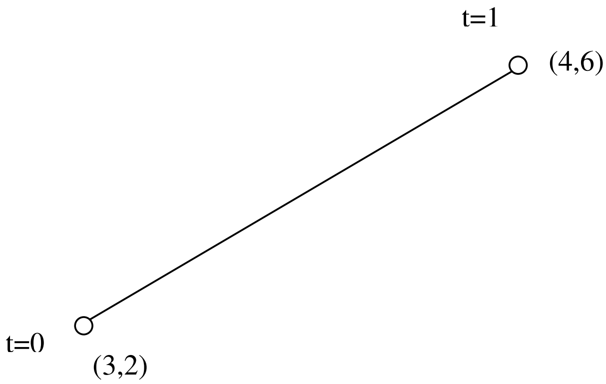
$$\int_{C_1} Re(z)dz = \int_0^1 t(1+i)dt = (1+i)\int_0^1 tdt = \frac{1}{2}(1+i). \tag{6.8}$$

$$\int_{C_2} Re(z)dz = \int_0^1 tdt = \int_0^1 idt = \frac{1}{2} + i. \tag{6.9}$$

Since the function $f(z) = Re(z)$ is not analytic. From (6.8) and (6.9) we see that the value of the integration depends not only on the end points but also on its geometric shape.

Example 6.3 Find the parametric equations for the line through the points (3,2) and (4,6) so that when t = 0 we are at the point (3,2) and when t = 1 we are at the point (4,6).

Solution:



We write symbolically:

$$(x, y) = (1 - t)(3, 2) + (t)(4, 6) = (3 - 3t + 4t, 2 - 2t + 6t) = (3 + t, 2 + 4t)$$

so that

$$x(t) = 3 + t \text{ and } y(t) = 2 + 4t.$$

$$x^2 + y^2 = 0 \tag{6.10}$$

- **Connected Set:** A set S of complex numbers is connected if, given any two points z and w in S, there is path in S having z and w as its end points.

- **Domain:** An open connected set of complex numbers is called a domain.

- **Simply Connected Domain:** A set S of complex numbers is simply connected if every closed path in S encloses only points of S.

- **Cauchy Integral Theorem:** If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D $\oint_C f(z)dz = 0$.

Example: $\oint_C e^z dz = 0$.

- **Cauchy Integral Formula:** If $f(z)$ is analytic in a simply connected domain D, then for any z_0 in D and any simple closed curve C that encloses z_0, $\oint_C \frac{f(z)}{z-z_0} = 2\pi i f(z_0)$, C : counterclockwise sense.

Example: $\oint_C \frac{e^z}{z} dz = 2\pi i$.

- **Liouville's Theorem:** If an entire function $f(z)$ is bounded in absolute value for all z, then $f(z)$ must be constant.

- **Morea's Theorem (Converse of Cauchy Integral Theorem):** If $f(z)$ is continuous in a simply connected domain D and if $\oint_C f(z)dz = 0$ for every closed path C in D, then $f(z)$ is analytic in D.

7 Improper Integral

An improper integral can be defined as,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

• We assume that $f(x)$ is a real rational function whose denomination is different from zero for all real x and is of degree at least two units higher than the degree of numerator. Then consider $\oint f(z)dz$. Where C is the contour given in above figure. Since $f(z)$ has no poles on the real axis by residue theorem.

$$\oint_c f(z)dz = 2\pi i \Sigma Res f(z)$$

$$\therefore L.H.S = \int_{-R}^{R} f(x)dx + \int_S f(z)dz$$

since,

$$\int_S f(z) \leq \frac{k}{R^2} \int_0^{\pi} dz = \frac{k\pi}{R} \to 0, as R \to \infty$$

thus,

$$\int_{-\infty}^{\infty} f(x)dx = \oint f(z)dz = 2\pi i \Sigma Res f(z)$$

Ex: Show that

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

Ans: Try this one.

FOURIER INTEGRALS:

The fourier integrals are,

$$\int_{-\infty}^{\infty} f(x)\cos sx dx, \int_{-\infty}^{\infty} f(x)\cos sx dx$$

same condition on $f(x)$ as earlier.

$$\int_{-\infty}^{\infty} f(x)\cos sx dx = -2\pi \Sigma Im Res[f(z)e^{isz}]$$

$$\int_{-\infty}^{\infty} f(x)\sin sx dx = 2\pi \Sigma Re Res[f(z)e^{isz}]$$

$\int_{-\infty}^{\infty} f(x)e^{ix}dx$ can be considered an improper integral.

Ex: Evaluate,

$$\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}dx$$

Ans: Try this one.

SIMPLE POLE ON REAL AXIS:

If $f(z)$ has a simple pole at $z = a$ on the real axis .Let C be the contour then ,

$$\lim_{r\to 0}\int_C f(z)dz = \pi i Res_{z=a} f(z)$$

Theorem:

Let $f(z) = \frac{h(z)}{g(z)}$, where h is continuous at z_0 and $h(z_0 \neq 0)$. Suppose g is differentiable at z_0 and has a simple zero there. Then f has a simple pole at z_0 and

$$Res(f, z_0) = \frac{h(z_0)}{g'(z_0)}$$

Ex:

Evaluate

$$\oint_\Gamma \frac{4iz-1}{\sin z}dz$$

Ans: Try this one.

Assignment

1. Discuss the boundedness of $\sin z$ and $\cos z$.
2. Find all roots of (i) $(1+i)^{\frac{1}{3}}$, (ii) $1^{\frac{1}{3}}$.
3. Solve the equations: (i) $z^2-(7+i)z+24+7i=0$, (ii) $z^4-(3+6i)z^2-8+6i$.
4. Find the values of Ref and Imf at $4i$, where $f = \frac{z-2}{z+2}$.
5. Discuss the continuity of $f(z) = \frac{Rez}{1+|z|}$.
6. Write the Cauchy Riemann equations in polar form.
7. Discuss the analyticity of the following functions.
   (i) $f(z) = z\cdot\bar{z}$ (ii) $f(z) = e^x(\cos y + i\sin y)$ (iii) $f(z) = \frac{Rez}{Imz}$ (iv) $f(z) = ln|z| + iArgz$.

8. Determine whether the following functions are Harmonic. If Yes, find the corresponding analytic functions $f(z) = u(x, y) + iv(x, y)$.
(i) $u = \ln z$ (ii) $v = -e^{-x} \sin y$ (iii) $v = (x^2 - y^2)^2$.

9. Determine a and b such that the given functions are harmonic and find its harmonic conjugate. (i) $u = ax^3 + by^3$ (ii) $u = e^{ax} \cos y$.

10. Find all points at which the following mappings are not conformal.
(i) $f(z) = z^2 + \frac{(}{1})(z^2)$ (ii)$f(z) = \frac{z^2+1}{z^2-1}$.

11. Find all solutions of the equations $\cos z = 3i$ and $\sin z = \cosh 3$.

12. Test the conformality of the mapping $f(z) = \cos z$. Find the conformal image of the region $0 < x < \pi,\ 0 < y < 1$.

13. Find the principal values of the following expressions.
(i) $(2i)^{2i}$ (ii) $(1 + i)^{-1+i}$ (iii) $(-3)^{3-i}$.

14. Find the linear fractional transformation that maps $\infty, 1, 0$ onto $0, 1, \infty$.

15. Find a linear fractional transformation that maps $|z| \leq 1$ onto $|w| \leq 1$ such that $z = \frac{i}{2}$ is mapped onto $w = 0$.

16. Find the fixed points of the map $f(z) = \frac{z+1}{z-1}$.

17. Show that
(a) the function $Log(z - i)$ is analytic every where except on the half line $y = 1$ $(x \leq 0)$;
(b) the function $\dfrac{Log(z + 4)}{z^2 + i}$ is analytic everywhere except at the points $\pm\frac{1-i}{\sqrt{2}}$ and on the portion $x \leq -4$ of the real axis.

18. Find the parametric representation $z = z(t)$ for
(a) For the upper half plane of $|z - 4 + 2i| = 3$,
(b) $|z + 3 - i| = 5$, counterclockwise.

19. Integrate $\displaystyle\int_C Rez^2 dz$, C the boundary of the square with vertices $0,\ i,\ 1 + i,\ 1$, clock wise.

20. Find a counter C such that the following integral gives the value 0.

(a) $\oint_C \frac{\cos z}{z^6 - z^2} dz$, (b) $\oint_C \frac{e^{\frac{1}{z}}}{z^2 + 9} dz$.

21. Evaluate

(a) $\oint_C \coth \frac{z}{2} dz$, C the circle $|z - \frac{1}{2}\pi i| = 1$, counterclockwise.

(b) $\oint_C \frac{2z^3 + z^2 + 4}{z^4 + 4z^2} dz$, C the circle $|z - 2| = 4$, clockwise.

22. Show that $\oint_C (z - z_1)^{-1}(z - z_2)^{-1} dz = 0$ for a simple closed path C enclosing z_1 and z_2, which are arbitrary.

23. Evaluate

(a) $\oint_C \frac{2z^3 - 3}{z(z - 1 - i)^2} dz$, C consists of $|z| = 2$ (counterclockwise) and $|z| = 1$ (clockwise).

(b) $\oint_C \frac{e^{z^2}}{(2z - 1)^2} dz$, C the circle $|z - i| = 2$, counterclockwise.

24. Test the convergency of the following series.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{(3i)^n n!}{n^n}$.

25. Find the center and the radius of convergence of the following power series.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} z^{2n+1}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n(n + 1)} z^{n-1}$ (c) $\sum_{n=2}^{\infty} n(n - 1)2^n z^{n^2}$.

26. Develop the given function in a Maclaurin series and find the radius of convergence.

(a) $e^{\frac{-z^2}{2}}$ (b) $\frac{1}{z+3i}$.

27. Develop $f(z) = \frac{2z-3i}{z^2-3iz-2}$ in a series (Taylor and Laurent) valid for

(a) $0 < |z| < 1$ (b) $1 < |z| < 2$ (c) $|z| > 2$ (d) $0 < |z + i| < 2$.

28. Determine the location and order of the zeros of the functions $\frac{z^2+1}{e^z-1}$ and $\tan \pi z$.

29. Determine the location and type of singularities of the functions $z^2 - \frac{1}{z^2}$ and $z^{-2} \sin^2 z$, including those at infinity.

30. Evaluate the following integrals (counterclockwise sense).

(a) $\oint_C \frac{e^z}{\cos z} dz$, $C: \ |z| = 3$ (b) $\frac{z \cosh \pi z}{z^4 + 13z^2 + 36} dz$, $C: \ |z| = \pi$.

31. Evaluate the following integrals using Residue Theorem.
(a) $\displaystyle\int_0^{\pi} \frac{d\theta}{k+\cos\theta}$ $(k > 1)$ (b) $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{13-12\cos\theta}d\theta.$

32. Evaluate the following integrals (counterclockwise sense).
(a) $\displaystyle\oint_C \frac{e^z}{\cos z}dz,\ C:\ |z|=3$ (b) $\displaystyle\frac{z\cosh\pi z}{z^4+13z^2+36}dz,\ C:\ |z|=\pi.$

33. Evaluate the following integrals using Residue Theorem.
(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 3x}{1+x^4}dx.$

34. State the following

(a) Maximum Modulus Principle

(b) Schwaz Lemma

(c) Residue Theorem

(d) Cauchy integral Theorem

(e) Argument principle

(f) Rouche's Theorem

(g) Conformal mapping

(i) Liouville's Theorem

(k) Fundamental Theorem of Algebra