1 Limit of function:

Let f be a real-valued function defined for all points in a neighbourhood N of a point c except possibly at the point c itself. Recall that any open set containing the element a is called the neighbourhood of a. In particular $N(a,\delta) = (a-\delta, a+\delta), \delta > 0$ is called the δ-neighbourhood of a and $N^*(a,\delta) = N(a,\delta) - \{a\} = (a-\delta, a) \cup (a, a+\delta)$ is called the deleted neighbourhood of a. We are assuming that f is defined in the deleted neighbourhood of a in the following definition of limit.

Definition 1 *The function $f : X \to \mathbb{R}$ is said to have a limit at $x = a$ (a may or may not belongs to X) if given $\varepsilon > 0$, there exists δ, depending upon a and ε, and there exists $L \in \mathbb{R}$ such that*

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon,$$

that is,

$$x \in (a-\delta, a) \cup (a, a+\delta) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon),$$

that is,

$$x \in N^*(a,\delta) \Rightarrow f(x) \in N(L,\varepsilon).$$

When this happens, we say that the limit L of f exists at $x = a$ and write it as

$$\lim_{x\to a} f(x) = L, \quad \text{or, } f(x) \to L \quad \text{as } x \to a.$$

This definition is called the $\varepsilon - \delta$ definition of limit.

One may observe that f need not be defined as a; even it is defined, it is not necessary that $f(a)$ be equal to L.

Definition 2 *The function f is said to tend to $+\infty$ as x tends to c (or in symbols, $\lim_{x\to c} f(x) = +\infty$) if for each $G > 0$ (however large), there exists a $\delta > 0$ such that*

$$f(x) > G, \quad \text{whenever } |x-c| < \delta.$$

Similarly, the function f is said to tend to $-\infty$ as x tends to c (or in symbols, $\lim_{x\to c} f(x) = -\infty$) if for each $G > 0$ (however large), there exists a $\delta > 0$ such that

$$f(x) < -G, \quad \text{whenever } |x-c| < \delta.$$

Definition 3 *The function f is said to tend to a limit l as x tends to ∞ (or in symbols, $\lim_{x\to\infty} f(x) = l$) if for each $\varepsilon > 0$, there exists a $k > 0$ such that*

$$|f(x) - l| < \varepsilon, \quad \text{whenever } x > k.$$

Definition 4 *The function f is said to tend to $+\infty$ as x tends to ∞ (or in symbols, $\lim_{x\to\infty} f(x) = \infty$) if for each $G > 0$ (however large), there exists a $k > 0$ such that*

$$f(x) > G, \quad \text{whenever } x > k.$$

1.1 Left hand and right hand limits

While defining the limit of a function $f(x)$ as x tends to c, we consider the values of $f(x)$ when x is very close to c. he values of x may be greater or less than c. If we restrict x to values less than c, then we say that x tends to c from below or from the left and write it symbolically as $x \to c-0$ or simply $x \to c-$. he limt of $f(x)$ with this restriction on x, is called the *left hand limit*. Similarly, if x takes only the values greater than c, then x is said to tend to c from above or from the right, and is denoted symbolically as $x \to c+0$ or $x \to c+$. The limit of $f(x)$ with this restriction on x, is called the *right hand limit*.

Definition 5 *A function f is said to tend to a limit l as x tends to c from the left, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$|f(x) - l| < \varepsilon, \quad \textit{whenever} \;\; c - \delta < x < c.$$

In symbols, we then write

$$\lim_{x \to c-} f(x) = l.$$

Definition 6 *A function f is said to tend to a limit l as x tends to c from the right, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$|f(x) - l| < \varepsilon, \quad \textit{whenever} \;\; c < x < c + \delta.$$

In symbols, we then write

$$\lim_{x \to c+} f(x) = l.$$

Note: We say $\lim\limits_{x \to c} f(x)$ exists if and only if both the limits (the left hand and the right hand) exists and are equal.

One-sided infinite limit may also be defined in the same way as above.

Example 1 *Let*

$$f(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a}, & x \neq a, \\ b, & x = a. \end{cases}$$

Then show that $\lim\limits_{x \to a} f(x) = 2a$, by using $\varepsilon - \delta$ definition.

Sol. This function defined on $\mathbb{R}$. Take $\varepsilon > 0$. The condition that $0 < |x - a| < \delta$, implies $x \neq a$. So we may write

$$\frac{x^2 - a^2}{x - a} = \frac{(x - a)(x + a)}{x - a} = x + a.$$

Now

$$|f(x) - 2a| = |(x + a) - 2a| = |x - a|$$

Since we need $|f(x) - 2a| < \varepsilon$ whenever $|x - a| < \delta$, clearly we can choose any δ such that $0 < \delta \leq \varepsilon$. Hence it follows from the definition that

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a.$$

Note that $2a = \lim\limits_{x \to a} \dfrac{x^2 - a^2}{x - a} \neq f(a) = b$, unless $b = 2a$.

Example 2 *Show that* $\lim_{x\to a} \dfrac{x^3-a^3}{x-a} = 3a^2$, *by using* $\varepsilon - \delta$ *definition.*

Sol. Let $\varepsilon > 0$. Take $x \neq a$. Then $f(x) = \frac{x^3-a^3}{x-a} = x^2 + xa + a^2$ and hence

$$\begin{aligned}\left|\frac{x^3-a^3}{x-a} - 3a^2\right| &= |(x^2-a^2) + a(x-a)| \\ &\leq |x-a||x+2a| \\ &\leq |x-a|(|x-a| + 3|a|)\end{aligned}$$

As we need $\delta > 0$ such that $|x-a| < \delta$, choosing first $\delta < 1$, the right hand side of the above inequality is less than or equal to $\delta(1+3|a|)$.
This gives an idea as to what δ can be choosen for a given $\varepsilon > 0$.
Choose $\delta = min\left(1, \dfrac{\varepsilon}{1+3|a|}\right)$ so that $\delta \leq \dfrac{\varepsilon}{1+3|a|}$.
Now

$$|f(x) - 3a^2| < \delta(1+3|a|) \leq \varepsilon.$$

This proves the result.

We observe that the choice of δ depends not only on ε, but also on a. Although the natural domain of f is $\mathbb{R} - \{a\}$.

Example 3 *Show that* $\lim_{x\to 2} \sqrt{4x+1} = 3$, *by using* $\varepsilon - \delta$ *definition.*

Sol. Let $\varepsilon > 0$ be given. Now

$$\begin{aligned}|\sqrt{4x+1} - 3| &= \left|\frac{(\sqrt{4x+1}-3)(\sqrt{4x+1}+3)}{\sqrt{4x+1}+3}\right| \\ &= \frac{4|x-2|}{\sqrt{4x+1}+3}\end{aligned}$$

Choose $\delta < 1$. So $|x-2| < \delta < 1$ implies $1 < x < 3$. Hence $\dfrac{1}{\sqrt{4x+1}+3} < \dfrac{1}{\sqrt{5}+3}$
So

$$|\sqrt{4x+1} + 3| \leq \frac{4\delta}{\sqrt{5}+3}.$$

In order that $\dfrac{4\delta}{\sqrt{5}+3} < \varepsilon$, we choose $\delta < \frac{(\sqrt{5}+3)\varepsilon}{4}$.
Now finally we choose $\delta = min(1, \frac{(\sqrt{5}+3)\varepsilon}{4})$ so that $\delta \leq \frac{(\sqrt{5}+3)\varepsilon}{4}$.
Hence $|\sqrt{4x+1} - 3| < \frac{4\delta}{\sqrt{5}+3} < \varepsilon$.
This proves the result.

Example 4 *Evaluate* $\lim_{x\to 0+} \dfrac{1}{1+\mathrm{e}^{-1/x}}$.

Sol. [As $x \to 0+$, we feel that $1/x$ increases indefinitely, $\mathrm{e}^{1/x}$ increases indefinitely. $\mathrm{e}^{-1/x}$ tends to 1; thus the required limit may be 1.]
We have to show that for a given $\varepsilon > 0, \exists$ a $\delta > 0$ such that

$$\left|\frac{1}{1+\mathrm{e}^{-1/x}} - 1\right| < \varepsilon, \quad \textit{whenever} \ \ 0 < x < \delta.$$

Now $\left|\dfrac{1}{1+\mathrm{e}^{-1/x}}-1\right| = \left|\dfrac{-\mathrm{e}^{-1/x}}{1+\mathrm{e}^{-1/x}}\right| = \dfrac{1}{1+\mathrm{e}^{1/x}} < \varepsilon$, when $1+\mathrm{e}^{1/x} > \frac{1}{\varepsilon}$ or $\frac{1}{x} > \log(\frac{1}{\varepsilon}-1)$

$$\Rightarrow 0 < x < \frac{1}{\log(1/\varepsilon - 1)}, \quad \text{for} \quad 0 < \varepsilon < 1.$$

Thus choosing $\delta = \frac{1}{\log(1/\varepsilon-1)}$, we see that if $0 < \varepsilon < 1$, $\left|\frac{1}{1+\mathrm{e}^{-1/x}}-1\right| < \varepsilon$, when $0 < x < \delta$.

Again when $\varepsilon \geq 1$, $\left|\frac{1}{1+\mathrm{e}^{-1/x}}-1\right| < \varepsilon \Rightarrow \mathrm{e}^{1/x} > \frac{1}{\varepsilon} - 1$, which is true for all values of x, so that for any $\delta > 0$ would work.

Thus for any $\varepsilon > 0$ we are able to find a $\delta > 0$ such that $\left|\dfrac{1}{1+\mathrm{e}^{-1/x}}-1\right| < \varepsilon$, when $0 < x < \delta$.

$\therefore \displaystyle\lim_{x\to 0+} \frac{1}{1+\mathrm{e}^{-1/x}} = 1.$

Example 5 *Prove that* $\displaystyle\lim_{x\to 0} x\sin\frac{1}{x} = 0.$

Sol. Now $\left|x\sin\dfrac{1}{x}\right| = |x|\left|\sin\dfrac{1}{x}\right| \leq |x|$

Thus choosing a $\delta = \varepsilon$, we see that $\left|x\sin\dfrac{1}{x}\right| < \varepsilon$, when $0 < |x| < \delta$.

$$\Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Example 6 *Show that* $\displaystyle\lim_{x\to 3}\frac{1}{(x-3)^4} = \infty.$

Sol. Let G be any positive number, however large.

Now $\left|\dfrac{1}{(x-3)^4}\right| > G$, or $\dfrac{1}{(x-3)^4} > G$, when $(x-3)^4 < \frac{1}{G}$ or when $0 < |x-3| < \frac{1}{G^{1/4}}$.

Choosing $\delta = \frac{1}{G^{1/4}}$, we get the required result.

Example 7 *Show that* $\displaystyle\lim_{x\to 1} 2^{1/(x-1)}$ *does not exist.*

Sol. We first consider the left hand limit. Let $\varepsilon > 0$ be given. Choosen a positive integer m such that $1/2^m < \varepsilon$.

Take $\delta = \frac{1}{m}$ and let x satisfy $1-\delta < x < 1$. Now $-\delta < (x-1) < 0$, and so $\frac{1}{x-1} < -\frac{1}{\delta} < 0$. Thus $|2^{1/(x-1)} - 0| = 2^{1/(x-1)} < 2^{-1/\delta} < 2^{-m} < \varepsilon$ and hence $\displaystyle\lim_{x\to 1-} 2^{1/(x-1)} = 0$.

Next, consider x to be on the right of 1.

Let $\delta > 0$ be arbitrary and choose a positive integer m_0 such that $\frac{1}{m_0} < \delta$. Then if $n \geq m_0$, $1+\frac{1}{n} \in (1, 1+\delta)$ and $2^{\frac{1}{1+\frac{1}{n}-1}} = 2^n$, which is unbounded. Therefore $\displaystyle\lim_{x\to 1} 2^{1/(x-1)}$ does not exist.

Example 8 *Find the right and the left hand limits of a function defined as follows*

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4, \\ 0, & x = 4. \end{cases}$$

Sol. When $x > 4, |x-4| = x-4$. $\therefore \lim\limits_{x\to 4+} f(x) = \lim\limits_{x\to 4+} \dfrac{|x-4|}{x-4} = \lim\limits_{x\to 4+} \dfrac{x-4}{x-4} = \lim\limits_{x\to 4+} 1 = 1.$
Again, when $x < 4, |x-4| = -(x-4)$. $\therefore \lim\limits_{x\to 4-} f(x) = \lim\limits_{x\to 4-} \dfrac{-(x-4)}{x-4} = \lim\limits_{x\to 4-} (-1) = -1.$
so that

$$\lim_{x\to 4+} f(x) \neq \lim_{x\to 4-} f(x).$$

Hence $\lim\limits_{x\to 4} f(x)$ does not exist.

Example 9 *If $\lim\limits_{x\to a} f(x)$ exists, prove that it must be unique.*

Sol. Let if possible, $f(x)$ tend to limits l_1 and l_2. Hence for any $\varepsilon > 0$ it is possible to choose a $\delta > 0$ such that
$|f(x) - l_1| < \varepsilon/2, \quad$ when $\quad 0 < |x-a| < \delta$.
$|f(x) - l_2| < \varepsilon/2, \quad$ when $\quad 0 < |x-a| < \delta$.
Now $\quad |l_1 - l_2| = |l_1 - f(x) + f(x) - l_2| \leq |l_1 - f(x)| + |f(x) - l_2| < \varepsilon$, when $\quad 0 < |x-a| < \delta$.
i.e., $|l_1 - l_2|$ is less than any positive number ε (however small) and so must be equal to zero. Thus $l_1 = l_2$.

Theorem 1 *(without proof) If f and g are two real valued functions defined on some neighbourhood of c such that $\lim\limits_{x\to c} f(x) = l$ and $\lim\limits_{x\to c} g(x) = m$ then*
(i) Let $\alpha \in \mathbb{R}$. We have $\lim\limits_{x\to c} \alpha f(x) = \alpha l$.
(ii) $\lim\limits_{x\to c}(f \pm g)x = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x) = l \pm m$.
(iii) $\lim\limits_{x\to c}(fg)x = \lim\limits_{x\to c} f(x). \lim\limits_{x\to c} g(x) = lm$.
(iv) $\lim\limits_{x\to c}(f/g)x = \lim\limits_{x\to c} f(x)/\lim\limits_{x\to c} g(x) = l/m, \quad$ provided $\; m \neq 0$.

Example 10 *Evaluate (i)* $\lim\limits_{x\to -1} \dfrac{(x+2)(3x-1)}{x^2+3x-2}, \quad$ *(ii)* $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2}{x}, \quad$ *(iii)* $\lim\limits_{x\to 0+} \dfrac{\sin x}{\sqrt{x}}$.

Sol. (i) $\lim\limits_{x\to -1} \dfrac{(x+2)(3x-1)}{x^2+3x-2} = \dfrac{\lim\limits_{x\to -1}(x+2). \lim\limits_{x\to -1}(3x-1)}{\lim\limits_{x\to -1} x^2+3x-2} = \dfrac{1.(-4)}{-4} = 1.$
(ii) $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2}{x} = \lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2}{x}.\dfrac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = \lim\limits_{x\to 0} \dfrac{1}{\sqrt{4+x}+2} = \dfrac{1}{4}.$
(iii) $\lim\limits_{x\to 0+} \dfrac{\sin x}{\sqrt{x}} = \left(\lim\limits_{x\to 0+} \dfrac{\sin x}{x}\right).\left(\lim\limits_{x\to 0+} \sqrt{x}\right) = 1.0 = 0.$

Example 11 *Evaluate* $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$.

Sol. Let us evaluate the left hand and right hand limits.
When $x \to 1-$, put $x = 1-h,\ h > 0$.
$\lim\limits_{x\to 1-} \dfrac{(x^2-1}{x-1} = \lim\limits_{h\to 0+} \dfrac{(1-h)^2-1}{-h} = \lim\limits_{h\to 0+} \dfrac{-h(2-h)}{-h} = \lim\limits_{h\to 0+} (2-h) = 2.$
Again when $x \to 1+$, put $x = 1+h,\ h > 0$.
$\lim\limits_{x\to 1+} \dfrac{(x^2-1}{x-1} = \lim\limits_{h\to 0+} \dfrac{(1+h)^2-1}{h} = \lim\limits_{h\to 0+} (2+h) = 2.$
So that both, the left hand and the right hand, limits exist and are equal. Hence limit of the given function exists and equals to 2.

Example 12 *Evaluate* $\lim\limits_{x\to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$.

Sol. Now when $x \to 0+, 1/x \to \infty, e^{-1/x} \to 0$ and $x \to 0-, 1/x \to -\infty, e^{1/x} \to 0$.

$\therefore \lim\limits_{x\to 0+} \dfrac{e^{1/x}}{e^{1/x}+1} = \lim\limits_{x\to 0+} \dfrac{1}{e^{-1/x}+1} = 1.$

and $\lim\limits_{x\to 0-} \dfrac{e^{1/x}}{e^{1/x}+1} = \dfrac{0}{1} = 0.$

so that the left hand limit not equal to the right hand limit. Hence $\lim\limits_{x\to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$ does not exist.

Example 13 *Find* $\lim\limits_{x\to 0} e^x sgn(x+[x])$, *where the signum function is defined as*

$$sgn(x) = \begin{cases} 1, & if\ x > 0, \\ 0, & if\ \ x = 0, \\ -1, & if\ x < 0, \end{cases} \quad \text{and } [x] \text{ means the greatest integer} \le x.$$

Sol. Now $\lim\limits_{x\to 0-} e^x sgn(x+[x]) = \lim\limits_{h\to 0+} e^{0-h} sgn(0-h+[0-h]) = \lim\limits_{h\to 0+} (-e^{-h}) = -1,$

$\lim\limits_{x\to 0+} e^x sgn(x+[x]) = \lim\limits_{h\to 0+} e^{0+h} sgn(0+h+[0+h]) = \lim\limits_{h\to 0+} e^h = 1.$

$\therefore \lim\limits_{x\to 0} e^x sgn(x+[x])$ does not exist.

Assignment 1

1. *Prove the following limits by using* $\varepsilon - \delta$ *definition:*

(i) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-2x} = 2,$

(ii) $\lim\limits_{x\to 6} \sqrt{x+3} = 3,$

(iii) $\lim\limits_{x\to a} x^n = a^n,$

(iv) $\lim\limits_{x\to a} f(x) = a^2$, *where* $f : \mathbb{R} \to \mathbb{R}$ *be given by* $f(x) = x^2$,

(v) $\lim\limits_{x\to a} f(x) = 1/a$, *where* $f : (0,\infty) \to \mathbb{R}$ *be given by* $f(x) = x^{-1}$.

2. *Evaluate the following limits (i)-(vi), if they exist:*

(i) $\lim\limits_{x\to 0} \dfrac{3x+|x|}{7x-5|x|},$ *(ii)* $\lim\limits_{x\to 1} \dfrac{1}{x-1}\left(\dfrac{1}{x+3} - \dfrac{2}{3x+5}\right),$

(iii) $\lim\limits_{x\to 0} \dfrac{1-2\cos x+\cos 2x}{x^2},$ *(iv)* $\lim\limits_{x\to\infty} \dfrac{e^x-e^{-x}}{e^x+e^{-x}},$

(v) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x(1-\cos x)},$ *(vi)* $\lim\limits_{x\to 1} \dfrac{1+\cos \pi x}{\tan^2 \pi x},$

(vii) Show that $\lim\limits_{x\to 0} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0,$

(viii) Show that $\lim\limits_{x\to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$ *does not exist,*

(ix) Show that $\lim\limits_{x\to 0} \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$ *does not exist,*

(x) If $\lim\limits_{x\to c} f(x) = l$ *then show that* $\lim\limits_{x\to c} |f(x)| = |l|$.

1.2 Limit of a function by sequential approach

Definition 7 *Let* $J \subset \mathbb{R}$ *be an interval. Let* $a \in J$. *Let* $f : J\backslash\{a\} \to \mathbb{R}$ *be given. Then* $\lim\limits_{x\to a} f(x) = l$ *iff for every sequence* $\{x_n\}$ *with* $x_n \in J\backslash\{a\}$ *with the property that* $x_n \to a$, *we have* $f(x_n) \to l$.

Theorem 2 *A function f tends to finite limit as x tends to c if and only if for every $\varepsilon > 0$ $\exists$ a neighbourhood $N(c)$ of c such that $|f(x_m) - f(x_n)| < \varepsilon$ for all $x_m, x_n \in N(c); x_m, x_n \neq c$.*

Similarly, a function f tends to a finite limit as x tends to ∞ if and only if for every $\varepsilon > 0$, there exists $G > 0$ such that $|f(x_m) - f(x_n)| < \varepsilon$, for all $x_m, x_n > G$.

Example 14 *Show that $\lim\limits_{x\to 0} \dfrac{1}{x} \sin \dfrac{1}{x}$ does not exist.*

Sol. Let $f(x) = \frac{1}{x} \sin \frac{1}{x}$. The function f is defined for every non-zero real number.
Now for each natural number n, let $x_n = \frac{2}{\pi(4n+1)}$, and so $f(x_n) = \frac{(4n+1)\pi}{2} \sin(2n\pi + \frac{\pi}{2}) = \frac{(4n+1)\pi}{2} \to \infty$ as $n \to \infty$.
$\therefore \lim\limits_{n\to\infty} f(x_n) = \infty$, when $\{x_n\} = \{\frac{2}{(4n+1)\pi}\}$ converges to zero.
Again, by taking $x_n = 1/n\pi$, we see that $f(x_n) = n\pi.0 = 0$ for every natural number n, and so $\lim\limits_{n\to\infty} f(x_n) \neq \infty$, when $\{x_n\} = \{1/n\pi\}$ converges to zero.
Therefore, $\lim\limits_{x\to 0} f(x)$ does not exist.

Example 15 *Find $\lim\limits_{x\to -\infty} x^2 sgn(\cos x)$.*

Sol. Here, $f(x) = x^2 sgn(\cos x)$. Let $x_n = -2n\pi$, so $\{x_n\} \to -\infty$, as $\to \infty$.
Now $f(x_n) = (-2n\pi)^2 sgn(\cos(-2n\pi)) = 4n^2\pi^2$, and so $\lim\limits_{n\to\infty} f(x_n) = \infty$, when $\{x_n\} = \{-2n\pi\} \to -\infty$.
$\therefore \lim\limits_{x\to -\infty} x^2 sgn(\cos x) = \infty$.
Again, taking $x_n = -(2n+1)\pi$, we see that $f(x_n) = [-(2n+1)\pi]^2 sgn(\cos(-(2n+1)\pi)) = -(2n+1)^2\pi^2$ and so $\lim\limits_{x\to -\infty} x^2 sgn(\cos x) = -\infty$.
Hence $\lim\limits_{x\to -\infty} x^2 sgn(\cos x)$ does not exist.

Assignment 2

(i) The function $f : \mathbb{R}\backslash\{0\} \to [-1, 1]$ defined by $f(x) = \sin(1/x)$ does not exist at $x = 0$ by using sequential approach.
(ii) Use sequential limit form to obtain the limit of $f(x) = x^3 + x^2 - 5$ at $x = a$.

Lemma 1 *(Sandwich Theorem) Let $J \subset \mathbb{R}$. Let f, g, h be defined on $J\backslash\{a\}$. Assume that*
(i) $f(x) \leq h(x) \leq g(x)$, for $x \in J, x \neq a$.
(ii) $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} g(x)$.
Then $\lim\limits_{x\to a} h(x) = l$.

2 Continuity

Let f be a real-valued function defined on an interval $J \subset \mathbb{R}$. We shall now consider the behaviour of f at points on J.

Definition 8 *($\varepsilon - \delta$ definition of continuity) Let $f : J \to \mathbb{R}$ be given and $a \in J$. We say that f is continuous at a if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in J$ and $|x - a| < \delta$ $\Rightarrow |f(x) - f(a)| < \varepsilon$.*

Definition 9 *A function $f(x)$ is said to be continuous at a point $c \in J$, if $\lim\limits_{x\to c} f(x)$ exists and the limit equals to the value of the function at $x = c$ (i.e., $\lim\limits_{x\to c} f(x) = f(c)$).*

A function f is said to be continuous in an interval J, if it continuous at every point of the interval.

A function is said to be discontinuous at a point $x = c$ of its domain, if it is not continuous at $x = c$. The point $x = c$ is called a point of discontinuity of the function.

Types of discontinuities:

(i) A function f is said to be have a removable discontinuity at $x = c$, if $\lim_{x \to c} f(x)$ exists but is not equal to the value $f(c)$ (which may or may not exist) of the function. Such a discontinuity can be removed by assigning a suitable value to the function at $x = c$.
(ii) A function f is said to have a discontinuity of the first kind at $x = c$, if $\lim_{x \to c-} f(x)$ and $\lim_{x \to c+} f(x)$ both exist but are not equal.
(iii) A function f is said to have a discontinuity of the first kind from the left at $x = c$, if $\lim_{x \to c-} f(x)$ exists but is not equal to $f(c)$.
(iv) A function f is said to have a discontinuity of the first kind from the right at $x = c$, if $\lim_{x \to c+} f(x)$ exists but is not equal to $f(c)$.
(v) A function f is said to have a discontinuity of the second kind at $x = c$, if neither $\lim_{x \to c-} f(x)$ nor $\lim_{x \to c+} f(x)$ exists.
(vi) A function f is said to have a discontinuity of the second kind from the left at $x = c$, if $\lim_{x \to c-} f(x)$ does not exist.
(vii) A function f is said to have a discontinuity of the second kind from the right at $x = c$, if $\lim_{x \to c+} f(x)$ does not exist.

Theorem 3 *(without proof) Let $f, g : J \to \mathbb{R}$ be continuous at a point $a \in J$. Let $\alpha \in \mathbb{R}$. Then the functions $\alpha f, |f|, f + g, f - g, fg$ are also continuous at $x = a$ and if $g(a) \neq 0$, then f/g is also continuous at $x = a$.*

Definition 10 *Let $f, g : J \to \mathbb{R}$ be a real-valued function and $a \in J$. Assume that f is continuous at a and g is continuous at $f(a)$. Then the composition function (gof) is also continuous at a.*

Definition 11 *Let $J \subset \mathbb{R}$. Let $f : J \to \mathbb{R}$ be a real-valued function and $a \in J$. We say that f is continuous at a if for every sequence $\{x_n\}$ in J with $x_n \to a$, we have $f(x_n) \to f(a)$.*

Example 16 *Discuss the continuity of the following functions:*
(i) $f(x) = 1/x$, (ii) $f(x) = \sin(1/x)$.

Sol. (i) The function $f(x) = 1/x$ has the natural domain $\mathbb{R}\backslash\{0\}$.
Here

$$\lim_{x \to 0+} f(x) = \infty \quad \text{and} \quad \lim_{x \to 0+} f(x) = -\infty.$$

Thus $f(x)$ has a discontinuity of the second kind at $x = 0$.
(ii) The function $f(x) = \sin(1/x)$ has the natural domain $\mathbb{R}\backslash\{0\}$.
It has been observed from Q. 1 in Assignment- 2, that it does not have a limit as $x \to 0$. In fact $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ do not exist as $f(x)$ oscillates between 1 and -1, as $x \to 0$. Thus $f(x)$ has a discontinuity of the second kind at $x = 0$.

Example 17 *Let*
$$f(x) = \begin{cases} x\sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$
Show that f is continuous at $x = 0$.

Sol. Since $|x\sin(1/x)| \leq |x|$, it follows from the Lemma 1 (Sandwich Theorem) that
$$\lim_{x\to 0} x\sin(1/x) = 0.$$
Thus $\lim_{x\to 0} f(x) = f(0)$.
Hence the function f is continuous at $x = 0$.

Example 18 *Discuss the continuity of*
$$f(x) = \begin{cases} 1 + x, & -\infty < x < 0, \\ 1 + [x] + \sin x, & 0 \leq x < \pi/2, \\ 3, & x \geq \pi/2. \end{cases}$$

Sol. Since $f(0) = 1 = \lim_{x\to 0-} f(x) = \lim_{x\to 0-}(1 + x) = 1 = \lim_{x\to 0+} f(x) = \lim_{x\to 0+}(1 + [x] + \sin x) = 1$.
It follows that f is continuous at $x = 0$.
Since $[x]$ is right continuous but not left continuous at $x = 1$, so also is f.
Again $f(\pi/2) = 3 = \lim_{x\to \pi/2+} f(x)$,
$\lim_{x\to\pi/2-} f(x) = \lim_{x\to\pi/2-}(1 + [x] + \sin x) = 1 + \lim_{x\to\pi/2-}[x] + \lim_{x\to\pi/2-}\sin x = 1 + 1 + 1 = 3.$
Hence f is also continuous at $\pi/2$. Thus f is continuous at every point of $\mathbb{R}$ except at $x = 1$.

Example 19 *Discuss the kind of discontinuity (if any) of the function defined as follows:*
$$f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x \neq 0, \\ 2, & x = 0. \end{cases}$$

Sol. Now $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \dfrac{x + x}{x} = 2$,
$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \dfrac{x - x}{x} = 0$,
and $f(0) = 2$.
Thus the function has discontinuity of the first kind from the right at $x = 0$.

Example 20 *Show that the function defined by* $f(x) = \begin{cases} \dfrac{xe^{1/x}}{1 + e^{1/x}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ *is continuous at $x = 0$.*

Sol. Now $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \dfrac{xe^{1/x}}{1 + e^{1/x}} = 0$,
$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \dfrac{x}{e^{-1/x} + 1} = 0$,
and $f(0) = 0$.
$\therefore \lim_{x\to 0} f(x) = 0 = f(0)$.
Thus the function is continuous at $x = 0$.

Example 21 *Discuss the continuity of*

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

at $x = 0$.

Sol. Now $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{\sin 2x}{2x}.2 = 2$, so that $\lim_{x\to 0} f(x) \neq f(0)$.
Hence the limit exists, but is not equal to the value of the function at the origin.
Thus the function has a removable discontinuity at $x = 0$.

Example 22 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be given by* $f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} - \mathbb{Q}. \end{cases}$ *Then* f *is not continuous at any point of* $\mathbb{R}$*. This is known as Dirichlet's function.*

Sol. Let $a \in \mathbb{Q}$ so that $f(a) = 1$.
Since in any interval there lie an infinite number of rational and irrational numbers, therefore for each positive integer n, we can choose an irrational number a_n such that $|a_n - a| < \frac{1}{n}$.
Thus the sequence $\{a_n\}$ converges to a.

But $f(a_n) = 0$ for all n and $f(a) = 1$, so that the sequence $f(a_n)$ does not converge to $f(a)$ (i.e., $\lim_{x\to\infty} f(a_n) \neq f(a)$). Thus we conclude that f is not continuous at all $a \in \mathbb{Q}$.

Next, let $b \in \mathbb{R} - \mathbb{Q}$. For each positive integer n we can choose a rational number b_n such that $|b_n - b| < \frac{1}{n}$. Thus the sequence $\{b_n\}$ converges to b.

But $f(b_n) = 1$ for all n and $f(b) = 0$, so that the sequence $f(b_n)$ does not converge to $f(b)$ (i.e., $\lim_{x\to\infty} f(b_n) \neq f(b)$). Thus we conclude that f is not continuous at all $b \in \mathbb{R} - \mathbb{Q}$.

$\therefore$ f is continuous nowhere on $\mathbb{R}$.

Assignment 3

1. *Discuss the continuity and classify the discontinuities, if any, of the following functions;*

(i) $f(x) = \begin{cases} x\sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$

(ii) $f(x) = \begin{cases} \dfrac{\sin(x-c)}{x-c}, & x \neq c, \\ 0, & x = c, \end{cases}$

(iii) $f(x) = \begin{cases} x^2, & x \geq 0, \\ x, & x < 0, \end{cases}$

(iv) $f(x) = \dfrac{1}{x-a}\, cosec \dfrac{1}{x-a}$,

(v) $f(x) = \begin{cases} (1+x)^{1/x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$

(vi) $f(x) = \begin{cases} \dfrac{e^{1/x}}{1 - e^{1/x}}, & x \neq 0, \\ 1, & x = 0, \end{cases}$

(vii) $f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ -x, & x \in \mathbb{R} - \mathbb{Q}. \end{cases}$

(viii) $f(x) = \begin{cases} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, & x \neq 0, \\ 1, & x = 0, \end{cases}$

(ix) $f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, & x \neq 0, \\ 0, & x = 0, \end{cases}$

(x) $f(x) = \begin{cases} 0, & x = 0, \\ (1/2) - x, & 0 < x < 1/2, \\ 1/2, & x = 1/2, \\ 3/2 - x, & 1/2 < x < 1, \\ 1, & x = 1, \end{cases}$

(xi) $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4}, & x \neq 2, \\ 3, & x = 2, \end{cases}$

(xii) $f(x) = \begin{cases} 2x, & x \leq 1, \\ x^2, & x > 1. \end{cases}$

3 Differentiability of functions

Definition 12 *Let J be an interval and $c \in J$. Let $f : J \to \mathbb{R}$. Then f is said to be differentiable at c, if there exists a real number α such that*

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \alpha. \tag{1}$$

It is sometimes useful to use the variable h for the increment $x - c$ and reformulate (1) as follows:

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \alpha. \tag{2}$$

The limit value α is called the derivative of the function f at $x = c$ and is denoted by $f'(c)$.

It should be noted that before examining the differentiability of f at c, it is necessary to ensure that f is defined in a neighbourhood of c.

Let J be an open interval and let $f : J \to \mathbb{R}$ be differentiable at every point of J, then we can define a new function $f' : J \to \mathbb{R}$ defined by

$$f'(x) = \lim_{y\to x} \frac{f(y) - f(x)}{y - x}.$$

Thus we get an operator D (say), which takes f to f' whenever f is differentiable. Thus $Df = f'$.

3.1 Left and right derivative

Definition 13 *Let $f : J \to \mathbb{R}$, $c \in J$. The function f is said have a left derivative at the point $x = c$ if there exists $m \in \mathbb{R}$ such that*

$$\lim_{x\to c-0} \frac{f(x) - f(c)}{x - c} = m,$$

and m is said to be left derivative of f at c and we denote it as $f'(c-0), f'(c-)$, $Lf'(c)$ or $D_-f(c)$.

Similarly, the function f is said have a right derivative at the point $x = c$ if there exists $m_0 \in \mathbb{R}$ such that

$$\lim_{x\to c+0} \frac{f(x) - f(c)}{x - c} = m_0,$$

and m_0 is said to be right derivative of f at c and we denote it as $f'(c+0), f'(c+), Rf'(c)$ or $D_+f(c)$.

NOTE: If a function is differentiable at $x = c$, then
(i) $D_-f(c)$ should exist.
(ii) $D_+f(c)$ should exist.
(iii) $D_-f(c) = D_+f(c) = Df(c)$.
Thus a function f is not differentiable if any one of the above requirements is not met.

In view of the definition of limit, the differentiability condition in (1) can be defined using $\varepsilon - \delta$ as follows:

Definition 14 *We say that f is differentiable at c if there exists $\alpha \in \mathbb{R}$ such that for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$x \in J \text{ and } 0 < |x - c| < \delta \Rightarrow |f(x) - f(c) - \alpha(x - c)| < \varepsilon|x - c|. \tag{3}$$

We say that f is differentiable on J if it is differentiable at each $c \in J$.

Example 23 *Let $f : J \to \mathbb{R}$ be a constant function, say, C. Then f is differentiable at $c \in J$ with $f'(c) = 0$ by using $\varepsilon - \delta$ definition.*

Sol. Consider the expression

$$\frac{f(x) - f(c)}{x - c} = \frac{C - C}{x - c} = 0.$$

This suggests that $\alpha = f'(c) = 0$. Now we will prove that $f'(c) = 0$ by using $\varepsilon - \delta$ definition. Let $\varepsilon > 0$ be given. Now let us try to estimate the error term:

$$|f(x) - f(c) - \alpha(x - c)| = |C - C - \alpha(x - c)| = |\alpha||(x - c)| = 0.$$

This suggests that we can choose any $\delta > 0$ for any $\varepsilon > 0$.

Let $\varepsilon > 0$ be given. Let $\delta > 0$ be arbitrary. We can estimate the error term:

$$|f(x) - f(c) - \alpha(x - c)| = 0 < \varepsilon|x - c|.$$

Thus if f is a constant function then it is differentiable in $\mathbb{R}$ with $f'(c) = 0$ for all $c \in J \subset \mathbb{R}$.

Example 24 *Let $f : J \to \mathbb{R}$ be given by $f(x) = ax + b$. Then f is differentiable on $\mathbb{R}$ with $f'(c) = a$, $c \in \mathbb{R}$ by using $\varepsilon - \delta$ definition.*

Sol. Let c be an arbitrary real number.
Consider the expression

$$\frac{f(x) - f(c)}{x - c} = \frac{(ax + b) - (ac + b)}{x - c} = \frac{a(x - c)}{x - c} = a.$$

This suggests that $\alpha = f'(c) = a$. Now we will prove that $f'(c) = a$ by using $\varepsilon - \delta$ definition. Let $\varepsilon > 0$ be given. Now let us try to estimate the error term:

$$|f(x) - f(c) - \alpha(x - c)| = |a(x - c) - a(x - c)| = 0.$$

This suggests that we can choose any $\delta > 0$ for any $\varepsilon > 0$.

Let $\varepsilon > 0$ be given. Let $\delta > 0$ be arbitrary. We can estimate the error term:

$$|f(x) - f(c) - \alpha(x - c)| = 0 < \varepsilon|x - c|.$$

Since c is arbitrary real number, f is differentiable on $\mathbb{R}$ and $f'(c) = a$.

Example 25 *Find the derivative of the function $f(x)$, where (i) $f(x) = \mathrm{e}^{\alpha x}$, $\alpha \in \mathbb{R}$, (ii) $f(x) = \log_a x$.*

Sol. (i) Given $f(x) = \mathrm{e}^{\alpha x}$. Then $f(x+h) - f(x) = \mathrm{e}^{\alpha(x+h)} - \mathrm{e}^{\alpha x} = \mathrm{e}^{\alpha x}(\mathrm{e}^{\alpha h} - 1)$.
Hence

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \mathrm{e}^{\alpha x} \lim_{h\to 0} \frac{\mathrm{e}^{\alpha h}-1}{h} \\
&= \mathrm{e}^{\alpha x} \lim_{h\to 0} \frac{1}{h}\left[\alpha h + \frac{(\alpha h)^2}{2!} + \cdots\right] = \alpha \mathrm{e}^{\alpha x}.
\end{aligned}$$

(ii) Given $f(x) = \log_a x$. Then $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\log_a(x+h) - \log_a x}{h} = \log_a\left(1+\dfrac{h}{x}\right)^{1/h}$.
Hence

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \log_a\left(1+\frac{h}{x}\right)^{1/h} \\
&= \log_a \lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h} \\
&= \log_a \mathrm{e}^{1/x} \\
&= (1/x)\log_a \mathrm{e}.
\end{aligned}$$

Example 26 *Discuss the differentiability of the following functions:*
(i) $f(x) = \begin{cases} x, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$ *at* $x = 1$ *and* *(ii)* $f(x) = x^2$ *on the interval* $[0, 1]$.

Sol. (i) Given that $f(x) = \begin{cases} x, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$
Now

$$D_- f(1) = \lim_{x\to 1-0} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{x-1}{x-1} = 1,$$

$$D_+ f(1) = \lim_{x\to 1+0} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{1-1}{x-1} = 0.$$

$\therefore \quad D_- f(1) \ne D_+ f(1)$.
Thus $f'(1) = \lim\limits_{x\to 1} \dfrac{f(x)-f(1)}{x-1}$ does not exist. Hence f is not differentiable at $x = 1$.
(ii) Given that $f(x) = x^2$. Let x_0 be any point on $(0, 1)$, then

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{x^2-x_0^2}{x-x_0} = \lim_{x\to x_0}(x+x_0) = 2x_0.$$

At the end points, we have

$$f'(0) = \lim_{x\to 0+0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{x^2}{x} = \lim_{x\to 0+} x = 0,$$

and

$$f'(1) = \lim_{x\to 1-0} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{x^2-1}{x-1} = \lim_{x\to 1-} x+1 = 2,$$

Thus the function f is derivable in the closed interval $[0, 1]$.

Theorem 4 *(without proof) A function which is differentiable at a point is continuous at the point; but not conversely.*

Theorem 5 *(without proof) Let $f, g : J \to \mathbb{R}$ be differentiable at $c \in J$. Then the following hold:*
(i) $f+g$ is differentiable at c with $(f+g)'(c) = f'(c) + g'(c)$,
(ii) αf is differentiable at c with $(\alpha f)'(c) = \alpha f'(c)$,
(iii) fg is differentiable at c with $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$,
(iv) f/g is differentiable at c with $(f/g)'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{\{g(c)\}^2}$, provided $g(c) \neq 0$,
(v) (gof) is differentiable at c with $(gof)'(c) = g'(f(c))f'(c)$.

Example 27 *The function $f(x) = |x|$ is continuous on $\mathbb{R}$ but not differentiable at $x = 0$.*

Sol. Given that $f(x) = |x|, \ \forall x \in \mathbb{R}$.
Now

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = 0 = f(0).$$

Thus f is continuous at $x = 0$.
We can easily check that f is continuous for all $x > 0$ and $x < 0$.
Thus f is continuous for all $x \in \mathbb{R}$.
But

$$D_+f(0) = \lim_{x\to 0+} \frac{|x| - 0}{x - 0} = \lim_{x\to 0+} \frac{x}{x} = \lim_{x\to 0+} 1 = 1.$$

and

$$D_-f(0) = \lim_{x\to 0-} \frac{|x| - 0}{x - 0} = \lim_{x\to 0-} \frac{-x}{x} = \lim_{x\to 0-} (-1) = -1.$$

Thus $D_+f(0) \neq D_-f(0)$.
Hence the function f is not differentiable at $x = 0$.

Example 28 *The function $f(x) = \begin{cases} x\sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$ is continuous but differentiable at $x = 0$.*

Sol. This function is continuous at $x = 0$ (see Example-17).
But since

$$\frac{f(x) - f(0)}{x - 0} = \frac{x\sin(1/x)}{x} = \sin(1/x),$$

it does not possess left or right limit at $x = 0$. Hence f is not differentiable at $x = 0$.

Example 29 *The function $f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$ is differentiable at $x = 0$ but f' is not continuous at $x = 0$ (i.e., $\lim_{x\to 0} f'(x) \neq f'(0)$).*

Sol. We have

$$Df(0) = f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2\sin(1/x)}{x} = \lim_{x\to 0} x\sin(1/x) = 0.$$

Thus f differentiable at $x = 0$ and $f'(0) = 0$.
If $x \neq 0$, then from the elementary calculus, we know that

$$f'(x) = 2x\sin(1/x) - \cos(1/x). \tag{4}$$

Clearly, $\lim_{x\to 0} f'(x)$ does not exist as $\cos(1/x)$ oscillates at $x = 0$ and therefore there is no possibility of $\lim_{x\to 0} f'(x)$ being equal to $f'(0)$.
Thus $f'(x)$ is not continuous at $x = 0$.
Note further from equation (4) that f' is not differentiable at $x = 0$.

Assignment 4 *Test the differentiability of the following functions at the points indicated.*

(i) $f(x) = \begin{cases} \sin x, & x \leq \pi/2, \\ 1 + (x - \pi/2)^2, & x > \pi/2, \end{cases}$ *at the point* $x = \pi/2$

(ii) $f(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$ *at the points* $x = -1$ *and* $x = 1$.

(iii) $f(x) = \begin{cases} e^{1/(x^2-1)}, & x < 1, \\ 0, & x \geq 1, \end{cases}$ *at the point* $x = 1$.

(iv) $f(x) = [x](x-1)$ *at* $x = 0$ *and* $x = 1$.

(v) $f(x) = |x| + |x - 1|$ *at* $x = 0$ *and* $x = 1$.

(vi) $f(x) = \begin{cases} e^{-1/x^2}, & x > 0, \\ 0, & x \leq 0, \end{cases}$ *at the point* $x = 0$.

(vii) $f(x) = \begin{cases} 2x - 3, & 0 \leq x \leq 2, \\ x^2 - 3, & 2 < x \leq 4, \end{cases}$ *at the points* $x = 2$ *and* $x = 4$.

(viii) $f(x) = \begin{cases} \dfrac{x(e^{1/x} - 1)}{e^{1/x} + 1}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ *at the point* $x = 0$.

(ix) $f(x) = \begin{cases} \dfrac{x(e^{1/x} - e^{-1/x})}{e^{1/x} + e^{-1/x}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ *at the point* $x = 0$.

(x) $f(x) = \begin{cases} x \tan^{-1}(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$ *at the point* $x = 0$.

Theorem 6 *Darboux Theorem (without proof)*
Let $f : [a, b] \to \mathbb{R}$ *be differentiable. Assume that* $f'(a) < \lambda < f'(b)$. *Then there exists* $c \in (a, b)$ *such that* $f'(c) = \lambda$. *(Thus though* f' *need not be continuous, it enjoys the intermediate value property.)*

Intermediate value theorem for derivatives- If a function f is derivable on a closed interval $[a, b]$ and $f'(a) \neq f'(b)$ and λ a number lying between $f'(a)$ and $f'(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = \lambda$.

Example 30 *Define* $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$

Then f is differentiable at all points including 0.
Here $f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$
It is easy to see that f' is not continuous (see Example-29).

According to Darboux theorem, f' enjoys the intermediate value property, even though it is not continuous.

4 Mean Value Theorems

On the basis of a certin amount of knowledge about the derivative of a function, mean value theorems enable us to get some information about the function itself. Sometimes, it is easier to tackle the derivative than the function. In this section we shall establish the so-called first derivative test.

Theorem 7 *Rolle's Theorem (without proof)*
Let $f : [a, b] \to \mathbb{R}$ be such that (i) f is continuous on $[a, b]$, (ii) f is differentiable on (a, b), and (iii) $f(a) = f(b)$. Then there exists at least one real point $c \in (a, b)$ such that $f'(c) = 0$.

The geometric interpretation is that there exists at least one point $c \in (a, b)$ such that slope of the tangent to the graph of f at c equals to zero. That is, the tangent at $(c, f(c))$ is parallel to the x-axis.
There is also an algebraic interpretation of Rolle's Theorem. If $f(a) = f(b) = 0$, then a and b are the zeros of $f(x)$ or the roots of the equation $f(x) = 0$. Thus, Rolle's Theorem says that if a and b are two roots of the equation $f(x) = 0$, then there exists at least one root $c \in (a, b)$ of the equation $f'(x) = 0$.

Example 31 *Show that the equation $10x^4 - 6x + 1 = 0$ has a root between 0 and 1.*

First determine the polynomial function f whose derivative is the polynomial, whose roots are being sought. So we take $f(x) = 2x^5 - 3x^2 + x$.

It is easily seen that f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Also $f(0) = 0 = f(1)$.

Hence using the Rolle's theorem, there exists $c \in (0, 1)$ such that $f'(c) = 10c^4 - 6c + 1 = 0$.

Example 32 *Prove that the function $f(x) = x^3 + x + k = 0$, k in any real constant, has exactly one real root.*

If it has two distinct roots, say c_1 and c_2, then $f(c_1) = 0$ and $f(c_2) = 0$.

Again, any polynomial function is continuous and differentiable on $\mathbb{R}$. Hence by Rolle's theorem there exists $c \in (c_1, c_2)$ such that $f'(c) = 3c^2 + 1 = 0$, which contradicts the fact that $c \in \mathbb{R}$.

Hence f has exactly one real root on $\mathbb{R}$.

Theorem 8 *Mean Value Theorem (MVT) or Langrange's Mean Value Theorem (without proof)*
Let $f : [a, b] \to \mathbb{R}$ be such that (i) f is continuous on $[a, b]$ and (ii) f is differentiable on (a, b). Then there exists at least one real number $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c).$$

There is a geometric interpretation of Mean Value Theorem. Under the given conditions, there exists $c \in (a, b)$ such that the slope of tangent to the graph of f at c equals that of the chord joining the two points $(a, f(a))$ and $(b, f(b))$.

Theorem 9 *Cauchy's form of Mean Value Theorem (without proof)*
Let $f, g : [a, b] \to \mathbb{R}$ be such that (i) f, g are continuous on $[a, b]$, (ii) f, g are differentiable on (a, b), and (iii) $g'(x) \neq 0$, for any $x \in (a, b)$. Then there exists at least one real number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Geometrically, Cauchy's form of MVT (Mean Value Theorem) means the following:
We look at the map $t \mapsto (g(t), f(t))$ from $\mathbb{J}$ to $\mathbb{R}^2$ as a parameterized curve in the plane. For example, $t \mapsto (\cos t, \sin t), t \in [0, 2\pi]$ is parameterization of a circle.

Then the slope of the chord joining the points $(g(a), f(a))$ and $(g(b), f(b))$ is $\frac{f(b)-f(a)}{g(b)-g(a)}$. The tangent vector to the parameterized curve at a point $(g(t_0), f(t_0))$ is $(g'(t_0), f'(t_0))$ and hence the tangent line at t_0 has the slope $f'(t_0)/g'(t_0)$. Thus Cauchy's mean value theorem says that there exists a point $c \in (a, b)$ such that slope $f'(c)/g'(c)$ of the tangent to the curve at c is equal to the slope of the chord joining the end points of the curve.

As observed in the remark on the geometric interpretation, if $g(x) = x$ in the Cauchy's mean value theorem, it reduces to Lagrange's mean value theorem.

Theorem 10 *Applicatons of Mean Value Theorem (MVT)*
Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b).
(i) If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b),
(ii) If $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant on (a, b),
(iii) If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b).

The mean value theorem is quite useful in proving certain inequalities. Here are some samples.

Example 33 *Show that* $\mathrm{e}^x > 1 + x$ *for all* $x \in \mathbb{R}\backslash\{0\}$.

Suppose $x > 0$. Consider the function $f(x) = \mathrm{e}^x$ on the interval $[0, x]$. Since e^x is differentiable on $\mathbb{R}$, we can apply mean value theorem to f on the interval $[0, x]$. Hence there exists $c \in (0, x)$ such that

$$\mathrm{e}^x - \mathrm{e}^0 = f'(c)(x - 0) = \mathrm{e}^c x.$$

Note that $f'(x) = \mathrm{e}^x > 1$ for all $x > 0$. So the displayed equation gives $\mathrm{e}^x - 1 = \mathrm{e}^c x > x$.

Similarly if $x < 0$, then consider the interval $[x, 0]$ and we can prove that $\mathrm{e}^x > 1 + x$.

Example 34 *Prove that* $\dfrac{y-x}{y} < \log\dfrac{y}{x} < \dfrac{y-x}{x}$, $0 < x < y$.

Let $0 < x < y$ and $f(x) = \log x$ on $[x, y]$. We know that $\log x$ is differentiable function on $x > 0$. Hence using the MVT, there exists $c \in (x, y)$ such that

$$\log y - \log x = f'(c)(y - x) \qquad \Rightarrow \log\frac{y}{x} = \frac{1}{c}(y - x).$$

Since $0 < x < c < y$, we have $\dfrac{1}{y} < \dfrac{1}{c} < \dfrac{1}{x}$. Hence we get

$$\frac{y-x}{y} < \frac{1}{c}(y - x) = \log\frac{y}{x} < \frac{y-x}{x}.$$

Example 35 *Prove that* $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, *for* $0 < \alpha < \theta < \beta < \pi/2$.

Let $f(x) = \sin x$ and $g(x) = \cos x$ for $x \in [\alpha, \beta]$.

$\therefore$ $f'(x) = \cos x$ and $g'(x) = -\sin x$.

Here the functions f and g are both continuous and differentiable. Therefore, by Cauchy's mean value theorem on $[\alpha, \beta]$,

$$\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}, \quad \alpha < \theta < \beta,$$

or,

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \quad \alpha < \theta < \beta.$$

Example 36 *Prove that* $\dfrac{x}{1+x} < \log(1+x) < x$ *for all* $x > 0$.

Let $f(x) = x - \log(1+x)$. Hence $f'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$. So by Theorem 10, f is strictly increasing. Since $f(0) = 0, f(x) > 0$ for $x > 0$. Thus $x > \log(1+x)$.

Similarly we consider the function $g(x) = \log(1+x) - \dfrac{x}{1+x}$ and show that $g(x) > 0$ for $x > 0$.

Example 37 *Show that* $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ *for* $0 < x < \pi/2$.

We have to show that $\dfrac{\tan x}{x} - \dfrac{x}{\sin x} > 0$, or $\dfrac{\sin x \tan x - x^2}{x \sin x} > 0$ for $0 < x < \pi/2$. Since $x \sin x > 0$ for $0 < x < \pi/2$, therefore it will sufficient to show that $\sin x \tan x - x^2 > 0$.

Let $f(x) = \sin x \tan x - x^2$, then for $0 < x < \pi/2$, $f'(x) = \sin x + \sin x \sec^2 x - 2x$. We cannot decide about the sign of $f'(x)$ mainly because of the presence of the $(-2x)$ term. Now the function $f'(x)$ is continuous and derivable on $(0, \pi/2)$.

$$\begin{aligned} \therefore \quad f''(x) &= \cos x + \cos x \sec^2 x + 2 \sin x \sec^2 x \tan x - 2 \\ &= (\sqrt{\sec x} - \sqrt{\cos x})^2 + 2 \tan^2 x \sec x > 0, \text{ for } \ 0 < x < \pi/2. \end{aligned}$$

Since the derivative $f''(x)$ of $f'(x)$ is positive, the function $f'(x)$ is an increasing function. Further since $f'(0) = 0$, therefore the function $f'(x) > 0$ for $0 < x < \pi/2$.

Again, since $f'(x) > 0$, $f(x)$ is an increasing function and because $f(0) = 0$, the function $f(x) > 0$, for $0 < x < \pi/2$.

Thus it follows that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ for $0 < x < \pi/2$.

Assignment 5 *Solve the following problems by using Rolle's theorem/ MVT;*
1. Prove that between any two real roots of $e^x \sin x = 1$*, there is at least one real root of* $e^x \cos x + 1 = 0$.
2. Show that the equation $\cos x = x^3 + x^2 + 4x$ *has exactly one root in* $[0, \pi/2]$.
3. Prove that the equation $x^3 - 3x^2 + b = 0$, $b \in \mathbb{R}$ *has at most one r 4. Let* $f : [2,5] \to \mathbb{R}$ *be continuous and be differentiable on* $(2,5)$*. Assume that* $f'(x) = (f(x))^2 + \pi$ *for all* $x \in (2,5)$*. True or false (with proper reason):* $f(5) - f(2) = 3$.
5. Apply Lagrange's mean value theorem for the function $f(x) = \log(1+x)$ *to show that*

$$0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1, \quad \text{for } \ \text{all } x > 0.$$

6. Establish the following inequalities:
(i) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3(1+x)} < \log(1+x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}, \quad x > 0,$
(ii) $\dfrac{x^2}{2(1+x)} < x - \log(1+x) < \dfrac{x^2}{2}, \quad x > 0,$
(iii) $\dfrac{x^2}{2} < x - \log(1+x) < \dfrac{x^2}{2(1+x)}, \quad -1 < x < 0,$
(iv) $(1+x) < e^x < 1 + xe^x$, *for all* x,
(v) $(1-x) < e^{-x} < 1 - x + \dfrac{x^2}{2}$, *for all* $x > 0$.

Theorem 11 *Taylor's Theorem (without proof)*
Assume that $f : [a, b] \to \mathbb{R}$ *be such that* $f^{(n)}$ *is continuous on* $[a, b]$ *and* $f^{(n+1)}(x)$ *exists on* (a, b). *Fix* $x_0 \in [a, b]$. *Then for each* $x \in [a, b]$ *with* $x \neq x_0$, *there exists* c *between* x *and* x_0 *such that*

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{(x - x_0)^k}{k!} f^{(k)}(x_0) + R_n, \tag{5}$$

where $R_n = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$.

The right-hand side of (5) is called the n-th order (or n-th degree) Taylor expansion of the function f at x_0. The expression $f(x) = f(x_0) + \displaystyle\sum_{k=1}^{n} \frac{(x - x_0)^k}{k!} f^{(k)}(x_0)$, is called the n-th degree Taylor polynomial of f at x_0. The term R_n is called the remainder term in the Taylor's expansion after $(n + 1)$ terms. The remainder term is the "error term" if we wish to approximate f near x_0 by the n-th order Taylor polynomial. If we assume that $f^{(n+1)}$ is bounded, say, by M on (a, b), then R_n goes to zero much faster than $(x - x_0)^n \to 0$.

Putting $x_0 = 0$ *in expression in (5) is called Maclaurin's Theorem with form of remainder.*

Example 38 *Show that tha function* $f(x) = \sin x,\ x \in [0, \pi/4]$ *is approximated by a polynomial* $\sin x = x - (x^3/6)$, *with an error less than* $1/400$.

The function f satisfies the condition of Taylor's theorem. Hence it can be expressed at $x_0 = 0$ by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + R_5,$$

where for some $c \in (0, x)$, and $R_5 = \dfrac{x^5}{5!} f^{(v)}(c)$. Thus

$$f(x) = x - \frac{x^3}{6} + \frac{x^5}{120} \cos c. \tag{6}$$

Now

$$|R_5| = \frac{x^5}{120} |\cos c| \leq \frac{x^5}{120} \leq \frac{1}{120} (\pi/4)^5 < \frac{1}{400}. \tag{7}$$

Hence it follow from (6) and (7) that

$$|f(x) - p(x)| < \frac{1}{400}, \quad \text{where} \quad p(x) = x - \frac{x^3}{6}.$$

Assignment 6 *1. If* $0 < x \leq 2$, *then prove that*

$$\log x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots .$$

2. Assuming the validity of expansion, show that
(i) $e^x \cos x = 1 + x - \dfrac{2x^3}{3!} - \dfrac{2^2 x^4}{4!} - \dfrac{2^2 x^5}{5!} + \cdots .$
(ii) $\log \sec x = \dfrac{1}{2} x^2 + \dfrac{1}{12} x^4 + \cdots .$
(iii) $\tan^{-1} x = \tan^{-1} \dfrac{\pi}{4} + \dfrac{x - \pi/4}{1 + \pi^2/16} - \dfrac{\pi(x - \pi/4)^2}{4(1 + \pi^2/16)^2} + \cdots .$
(iv) $\sin(\dfrac{\pi}{4} + \theta) = \dfrac{1}{\sqrt{2}} \left(1 + \theta - \dfrac{\theta^2}{2!} - \dfrac{\theta^3}{3!} + \cdots \right).$

References

[1] A. Kumar and S. Kumaresan, *A basic course in Real Analysis*, CRC Press, 2014.

[2] G. Das and S. Pattanayak, *Fundamentals of Mathematical Analysis*, Tata McGraw Hill Publication, 2010.

[3] S.C. Malik and S. Arora, *Mathematical Analysis*, New Age International Publication, 2008.

Lecture Notes

Mathematics-I, B.Tech, First Semester

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UNIT-II

1 Linear Algebra

1.1 Matrix

A matrix is a rectangular array of numbers (or functions) enclosed in brackets.

Example 1.1.

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \end{array}$$

or

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

or

$$\begin{Bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{Bmatrix}$$

or

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

or

$$\begin{Bmatrix} f_1(x) & f_2(x) & f_3(x) & f_4(x) \end{Bmatrix}$$

Vector:
A vector is a matrix that has only one row- we call row matrix- or only one column we called column matrix. Entries of matrix are called components.

Example 1.2.

$$\begin{Bmatrix} a_1 & a_2 & a_3 & a_4 \end{Bmatrix}$$

or

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}$$

Transposition:
Given a matrix

$$A=[a_{jk}]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & . & . & a_{mn} \end{bmatrix}$$

. The transpose of A is written as A^T is an $n \times m$ matrix defined as follows

$$A^T=[a_{kj}]=\begin{bmatrix} a_{11} & a_{21} & a_{31} & . & . & . & a_{m1} \\ a_{12} & a_{22} & a_{32} & . & . & . & a_{2n} \\ a_{13} & a_{23} & a_{33} & . & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ a_{1n} & a_{2m} & a_{3m} & . & . & . & a_{mn} \end{bmatrix}$$

Symmetric and Skew symmetric, Equal matrix:-
A matrix A is called symmetric, if $A = A^T$ and is called skew symmetric matrix if $A^T = -A$, two matrices are equal if they have same size and corrsponding entries are equal.
Matrix Addition:- Addition of matrices A=$[a_{jk}]$ and B=$[a_{il}]$ is defined only when both are of same size, their sum $A+B$ is obtained by adding the corrsponding entries.

Example 1.3.

$$\begin{bmatrix} 3 & 4 & -1 \\ 4 & -5 & 7 \\ 0 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -1 \\ 3 & -5 & 2 \\ 0 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ 7 & -10 & 9 \\ 0 & 6 & -3 \end{bmatrix}$$

Scalar Multiplication: The product of any $m \times n$ matrix $A = [a_{jk}]$ and any scalar c written as cA, is the $m \times n$ matrix $cA = [ca_{jk}]$
Matrix multiplication: Let $A = [a_{jk}]$ is an $m \times n$ matrix and $B = [a_{jk}]$ is an $r \times p$ matrix, then product $C = AB$ is defined if and only if $r = n$, with $C = [c_{jk}]$

where

$$c_{jk} = \sum_{i=1}^{n} a_{ji}b_{ik} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots a_{jn}b_{nk} \quad j = 1.2...m \text{ and } k = 1.2...p$$

Example 1.4.

$$AB = \begin{bmatrix} 4 & 3 \\ 7 & 2 \\ 9 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 38 \\ 16 & 47 \\ 18 & 45 \end{bmatrix}$$

Some Properties of matrix multiplication

1. $(kA)B = k(AB) = A(kB)$, where K is any scalar.
2. $A(BC) = (AB)C$.
3. $(A + B)C = AC + BC$.
4. $AC = AD \nRightarrow C = D$, even when $A \neq 0$.
5. Matrix multiplication is not commutative in general.

Special Matrices: We now list here some of the important matrices: **Trangular Matrices**

Upper triangular matrices are square matrices that can have nonzero entries only on and above the main diagonal where as any entry below the diagonal must be zero. Similarly lower triangular matrices can have nonzero entries only on and bellow the main diagonal.

Example 1.5.
Upper and Lower triangular matrices

$$\begin{pmatrix} 1 & 5 \\ 0 & -5 \end{pmatrix}, \quad \begin{pmatrix} 7 & 3 & 2 \\ 0 & 6 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 7 & 8 & 0 \\ 8 & -2 & 1 \end{pmatrix}$$

The first two matrices are upper triangular whereas the last one is the lower triangular matrix.

Diagonal matrices The square matrices whose main diagonal entries are nonzero are called diagonal matrices.

Example 1.6.
diagonal matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \begin{pmatrix} 14 & 0 & 0 \\ 0 & 86 & 0 \\ 0 & 0 & 57 \end{pmatrix}$$

Transpose of a product: The transpose of a product equals the product of the transposed factors, taken in reverse order,

$$(AB)^T = B^T A^T.$$

Inner product of vectors: If a is a row vector and b is column vector both with n components then the inner product or dot product of $a$and b is defined by

$$a.b = [a_1, a_2, \cdots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i b_i = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

Homework

$$a = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{pmatrix}, \quad d = \begin{pmatrix} 4 & 3 & 0 \end{pmatrix}$$

Calculate the following products.

1. $Ba, \ a^T B, \ aB.$

2. $C^2, \ C^T C, \ CC^T.$

3. $a^T d, \ B^T B, \ da, \ ad.$

4. $Ca, \ C^2 a, \ C^3 a.$

Linear system of equations, Gauss Elimination: A linear system of m-equations in n- unknowns $x_1, x_2 ... x_n$, is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$

$$..$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m \qquad (1)$$

A solution of (1) is a set of numbers $x_1, x_2 ... x_n$, that satisfy all the m equations.

Matrix form of linear equation: $Ax = b$, where A is the coefficient matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & . & . & a_{mn} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ . \\ b_m \end{bmatrix}$$

If all $b_i = 0$ then (1.1) is called homogeneous system. If at least one $b_i \neq 0$ then the system is called non homogeneous.
The matrix

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & . & . & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & . & . & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & . & . & a_{3n} & b_3 \\ . & . & . & . & . & . & . & \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & . & . & a_{mn} & b_m \end{bmatrix}$$

is called **augmented** matrix.

Theorem 1.7. *Row equivalent linear systems have the same sets of solution.*

Gauss Elimination Method:

Example 1.8.

Solve the linear system.

$$- x_1 + x_2 + 2x_3 = 2$$

$$3x_1 - x_2 + x_3 = 6$$

$$- x_1 + 3x_2 + 4x_3 = 4$$

Solution 1.9.

$$\begin{bmatrix} -1 & 1 & 2 & | & 2 \\ 3 & -1 & 1 & | & 6 \\ -1 & 3 & 4 & | & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 & | & 2 \\ 3 & 2 & 7 & | & 12 \\ 0 & 2 & 2 & | & 2 \end{bmatrix} R_2 \leftarrow R_2 + 3R_1$$

$$R_2 \leftarrow R_3 - 3R_1$$

$$\begin{bmatrix} -1 & 1 & 2 & | & 2 \\ 0 & 2 & 7 & | & 12 \\ 0 & 0 & -5 & | & -10 \end{bmatrix} R_3 \leftarrow R_2 - 3R_2$$

The above row equivalent form gave a set of equation as follows

$$- x_1 + x_2 + 2x_3 = 2$$

$$2x_2 + 7x_3 = 12$$

$$- 5x_3 = 10$$

Solving the above system of equation we get
$x_1 = 2,\ x_2 = -1,\ x_3 = 1$

Home work Solve the following systems by the Gauss-elimination method.

1.
$$6x + 4y = 2$$
$$3x - 5y = -34.$$

2.
$$0.4x + 1.2y = -2$$
$$1.7x - 3.2y = 8.1.$$

3.
$$13x + 12y = -6$$
$$-4x + 7y = -72$$
$$11x - 13y = 157.$$

4.
$$1.3x - 9.1y + 11.7z = 0$$
$$-0.9x + 6.3y - 8.1z = 0$$

2 Rank of a matrix, Linear Independence:

Linear Independence and Dependence of Vectors Given any set of m vectors (with the same number of components), a linear combination of these vectors is an expression of the form

$$c_1a_1 + c_2a_2 + \cdots + c_ma_m$$

where $c_1, c_2, \cdots, c_m$ are any scalars. Now consider the equation

$$c_1a_1 + c_2a_2 + \cdots + c_ma_m = 0 \tag{2}$$

If all the scalars c_j are zero, then our vectors are said to form a linearly independent set or, more briefly, we call them linearly independent. Otherwise, if (2) also holds with scalars not all zero, we call these vectors linearly dependent. This means that we can express at least one of the vectors as a linear combination of the other vectors.

Example 2.1.

Linear Independence and Dependence The three vectors

$$a_1 = [3, 0, 2, 2], \ \ a_2 = [-6, 42, 24, 54] \ \ a_3 = [21, -21, 0, -15]$$

are linearly dependent because

$$a_3 = 6a_1 - \frac{1}{2}a_2.$$

Rank of a Matrix

Definition 2.2. The rank of a matrix A is the maximum number of linearly independent row vectors of A. It is denoted by rank A.

Theorem 2.3. *Row-equivalent matrices have the same rank.*

Theorem 2.4. *The rank of A equals the maximum number of linearly independent column vectors of A. Hence A and A^T have the same rank.*

Theorem 2.5. *Consider p vectors each having n components. If $n < p$ then these vectors are linearly dependent.*

Example 2.6.

Find the rank of the matrix $A = \begin{pmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{pmatrix}$

Solution 2.7. The given matrix A has rank 2 because first two row vectors are linearly independent, whereas all three row vectors are linearly dependent.

Home work Find the rank of the matrix.

1.

$$\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{pmatrix}, \begin{pmatrix} 4 & 3 & 0 \end{pmatrix}$$

2.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \begin{pmatrix} 14 & 0 & 0 \\ 0 & 86 & 0 \\ 0 & 0 & 57 \end{pmatrix}$$

3.

$$\begin{pmatrix} 1 & 5 \\ 0 & -5 \end{pmatrix}, \begin{pmatrix} 7 & 3 & 2 \\ 0 & 6 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

4.

$$\begin{pmatrix} 1 & 0 & 0 \\ 7 & 8 & 0 \\ 8 & -2 & 1 \end{pmatrix}$$

3 Vector spaces and subspaces

Fields of scalars

Definition A field of scalars (or just a field) consists of a set F whose elements are called scalars, together with two algebraic operations, addition + and multiplication ., for combining every pair of scalars $x,\ y \in F$ to give new scalars $x + y,\ x.y \in F$

Definition A vector space over a field of scalars F consists of a set V whose elements are called vectors together with two algebraic operations, + (addition of vectors) and (multi- plication by scalars). Vectors will usually be denoted with boldface symbols such as v ,The operations + and . are required to satisfy the following rules, which are sometimes known as the vector space axioms.

Associativity: For $u,\ v,\ w \in V$and$s,\ t \in F$, $(u + v) + w = u + (v + w)$; $(s.t).v = s.(t.v)$;

Zero and unity: There is a unique element $0 \in V$such that for $v \in V$, $v + 0 = v = 0 + v$; and multiplication by $1 \in F$satisfies $1.v = v$:

Distributivity: For $s,\ t \in F$ and $u,\ v \in V$, $(s + t).v = s.v + t.v$; $s.(u + v) =$

$s.u + s.u$:

Commutativity: For $u; v2V$, $u + v = v + u$:

Additive inverses: For $v \in V$ there is a unique element $-v \in V$ for which $v + (-v) = 0 = (-v) + v$

Definition Sub Space: Let V be a vector space over a field of scalars F. Suppose that the subset W of V is non-empty and is closed under addition and multiplication by scalars i.e., for $s \in F,\ u,\ v \in W\ u + v \in W$ and $su \in W$ thus it forms a vector space over F. Then W is called a (vector) subspace of V .

The maximum number of linearly independent vectors in V is called the dimension of V and is denoted by dim V. A linearly independent set in V consisting of a maximum possible number of vectors in V is called a basis for V. In other words, any largest possible set of independent vectors in V forms basis for V. That means, if we add one or more vector to that set, the set will be linearly dependent. Thus, the number of vectors of a basis for V equals dim V. The set of all linear combinations of given vectors $a_1, a_2, \cdots, a_n$ with the same number of components is called the span of these vectors. Obviously, a span is a vector space. If in addition, the given vectors $a_1, a_2, \cdots, a_n$ are linearly independent, then they form a basis for that vector space.

4 Inverse of a matrix, Gauss Jordan Elimination:

In this section we consider square matrices exclusively. The inverse of an $n \times n$ matrix is denoted by A^{-1} and is an $n \times n$ matrix such that

$$AA^{-1} = A^{-1}A = I$$

where I is the unit matrix. If A has an inverse, then A is called a nonsingular matrix. If A has no inverse, then A is called a singular matrix. If A has an inverse, the inverse is unique. Indeed, if both B and C are inverses of A, then $AB = I$ and $CA = I$ so that we obtain the uniqueness from

$$B = IB = (CA)B = C(AB) = CI = C.$$

We prove next that A has an inverse (is nonsingular) if and only if it has maximum possible rank n. The proof will also show $Ax = b$ that implies $x = A^{-1}b$ provided

A^{-1} exists, and will thus give a motivation for the inverse as well as a relation to linear systems.

Theorem 4.1. *Existence of the Inverse*
The inverse A^{-1} of an $n \times n$ matrix A exists if and only if rank $A = n$, thus if and only if $|A| \neq 0$. Hence A is nonsingular if rank $A = n$ and is singular if rank $A < n$.

Determination of the Inverse by the GaussJordan Method To determine the inverse of a nonsingular matrix A, we can use a variant of the Gauss elimination called the GaussJordan elimination. The idea of the method is as follows. Using the matrix A, we form n linear systems

$$Ax_1 = e_1, Ax_2 = e_2, \cdots Ax_n = e_n.$$

where the vectors $e_1, e_2, \cdots e_n$ are the columns of the unit matrix I. These are n vector equations in the unknown vectors $x_1, \cdots, x_n$. We combine them into a single matrix equation $AX = I$ with the unknown matrix X having the columns $x_1, x_2, \cdots x_n$. Correspondingly, we combine the n augmented matrices $[A, e_1], \cdots [A, e_n]$ into one wide $n \times 2n$ augmented matrix $\widetilde{A} = [AI]$. Now multiplication of $AX = I$ by A^{-1} from the left gives $X = A^{-1}I = A^{-1}$. Hence, to solve $AX = I$ for X, we can apply the Gauss elimination to $\widetilde{A} = [AI]$. This gives a matrix of the form $[UH]$ with upper triangular U because the Gauss elimination triangularizes systems. The GaussJordan method reduces U by further elementary row operations to diagonal form, in fact to the unit matrix I. This is done by eliminating the entries of U above the main diagonal and making the diagonal entries all 1 by multiplication.

Exercises Calculate the inverse by the Gauss-Jordan Elimination

1.
$$\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

2.

$$\begin{pmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{pmatrix}$$

3.

$$\begin{pmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

UNIT- III

5 First order differential equation:

Definition 5.1. A differential equation is an equation involving derivatives of one or more dependent variable with respect to one or more independent variable.

For example,

$$y'' + 4y = 0$$

$$y' + xy = x^2.$$

$$y''' + x^4 y'' + y'^4 + y = \cos x.$$

Definition 5.2. An ordinary differential equation is one which involves only one independent variable, so that all the derivative occurring in the differential equation are ordinary derivatives.

Solution of a differential equation: When we say $x = 1$ is a solution of the algebraic equation $x^2 = 1$ we mean that when $x = 1$ is substituted in the equation the equality will hold. Similarly we say that $y = x^2$ is a solution of the differential equation $\frac{dy}{dx} = 2x$ since if we put $y = x^2$ in the above equation the equality holds. Thus we give the formal definition of the solution of a general ordinary differential equation of nth order

$$F(x, y, y', y'', \cdots y^{(n)}) = 0 \tag{3}$$

as follows

Definition 5.3. Let $y = f(x)$ be a real valued function on a interval I, then f is called an explicit solution of the differential equation (3) if the substitution $y = f(x)$ reduces to an identity in x on I, i.e., if

$$F(x, f(x), f'(x), \cdots, f^{(n)}(x)) = 0$$

for every x in I.

Definition 5.4. A relation $g(x, y) = 0$ is called an implicit solution of the differential equation (3) on I if this relation defines at-least one real function of x, $y = \phi(x)$ on I such that ϕ is an explicit solution of (3) on I.

Example 5.5.
Verification of solution:

Verify that $y = x^2$ is a solution of $xy' = 2y$ for all x.
Indeed by substituting $y = x^2$ and $y' = 2x$ into the equation we obtained $xy' = x(2x) = 2x^2 = 2y$, an identity in x.

Example 5.6.
The relation

$$x^2 + y^2 = 1 \tag{4}$$

is implicit solution of the differential equation

$$x + y\frac{dy}{dx} = 0 \tag{5}$$

on the interval $I : -1 < x < 1$. To verify this, first we see that the relation (4) defines two real functions.

$$\phi_1(x) = +\sqrt{1 - x^2}$$
$$and \ , \phi_1(x) = -\sqrt{1 - x^2}, \ \ for \ \ x \in I : -1 < x < 1.$$

Next we see that the real function ϕ_1 is an explicit solution of (5), for substituting $y = \phi_1(x) = \sqrt{1 - x^2}$ and $y' = \phi_1'(x) = \frac{-x}{\sqrt{1-x^2}}$ In (5) we obtain $x + \sqrt{1 - x^2}\frac{-x}{\sqrt{1-x^2}} = 0$ which is an identity for all $x \in I$. Thus the relation (4) is an implicit solution of (5).

Generally it is difficult to solve first order ordinary differential equations $\frac{dy}{dx} = f(x, y)$ in the sense that no formulae exist for obtaining its solution in all cases. However their are certain standard types of first order differential equations of first degree for which routine methods of solution are available. In this unit we shall discuss a few of these types.

6 Separable differential equation:

Differential equation of the form

$$g(y)dy = f(x)dx$$

are called equations with separated variables, the solutions of which are obtained by direct integration. Thus its solution is given by

$$\int g(y)dy = \int f(x)dx + c.$$

Example 6.1.

Solve the differential equation

$$x(1+y^2)dx - y(1+x^2)dy = 0.$$

Solution 6.2. Separating the variables by dividing the product of $(1+y^2)(1+x^2)$ we get

$$\frac{ydy}{1+y^2} = \frac{xdx}{1+x^2}.$$

By integrating on both sides we obtain the general solution $1+y^2 = c(1+x^2$.

Example 6.3.

Solve the differential equation

$$9yy' + 4x = 0.$$

Solution 6.4. By separating the variables we have $9ydy = -4xdx$. On integration, this yields

$$\frac{x^2}{9} + \frac{y^2}{4} = c.$$

The solution represents a family of ellipses.

Exercise: Solve the following differential equation

1. $yy' + 25x = 0$.
2. $y' + 3x^2y^2 = 0$.
3. $y' = \frac{x^2+y^2}{xy}$.
4. $xydx + (x+1)dy = 0$.
5. $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$.
6. $(x+y)dx + dy = 0$.

7 Exact differential equations and Integrating Factor:

Definition 7.1. The differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{6}$$

is called exact if there exist a function $u(x, y)$ such that $du(x, y) = M(x, y)dx + N(x, y)dy$

Theorem 7.2. *The differential equation*

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

A working rule to find out the solution of the exact differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is as follows:

$$\int_{\text{y as constant}} M(x, y)dx + \int_{\text{only those terms which do not contain x}} N(x, y)dy = c$$

Example 7.3.
Solve

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0.$$

Solution 7.4. Here $M(x, y) = (x^3 + 3xy^2)$ and $N(x, y) = (3x^2y + y^3)$. Thus

$$\frac{\partial M}{\partial y} = 6xy, \ \frac{\partial N}{\partial x} = 6xy.$$

Since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

the given equation is exact. Hence the solution of the differential equation is

$$\int (x^3 + 3xy^2)dx + \int (3x^2y + y^3)dy = c.$$

In the first integration we treat y as constant and in the second we will integrate only those terms which do not contain x. That implies

$$\frac{1}{4}(x^4 + 6x^2y^2 + y^4) = c$$

is the desired solution.

Example 7.5.
Test the equation

$$e^y dx + (xe^y + 2y)dy = 0$$

for exactness, and solve it if it is exact.

Solution 7.6. Here $M(x,y) = e^y$ and $N(x,y) = xe^y + 2y$. Therefore

$$\frac{\partial M}{\partial y} = e^y, \ \textit{and} \ \frac{\partial N}{\partial x} = e^y.$$

Since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

so the given differential equation is exact. Now,

$$\int e^y dx + \int (0 + 2y) dy = c.$$

That implies

$$xe^y + y^2 = c$$

is the desired solution.

Exercise: Solve the following differential equation

1. $3y^2 dx + x dy = 0, \ y(1) = \frac{1}{2}$
2. $2 \sin 2x \sinh y dx - \cos 2x \cosh y dy = 0, \ \ y(0) = 1.$
3. $\frac{(3-y)}{x^2} dx + \frac{(y^- 2x)}{xy^2} dy = 0, \ \ y(-1) = 2.$
4. $2xydy = (x^2 + y^2) dx, \ \ y(1) = 2.$

8 Integrating Factors

Definition 8.1. An integrating factor is a function when multiplied by it, the left hand side of the equation (6) becomes an exact differential equation.

How to find integrating factor:

Theorem 8.2. *If*

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x),$$

a function of x alone, then $e^{\int f(x)dx}$ is an integrating factor of the equation (6).

Theorem 8.3. *If*

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(y),$$

a function of y alone, then $e^{-\int f(y)dy}$ is an integrating factor of the equation (6).

Example 8.4.

Find an integrating factor and solve the initial value problem:

$$2\sin y^2 dx + xy\cos y^2 dy = 0, \;\; y(2) = \sqrt{\frac{\pi}{2}}.$$

Here

Solution 8.5. Here $M(x, y) = 2\sin y^2$ and $N(x, y) = xy\cos y^2$. Now

$$\frac{\partial M}{\partial y} = 4y\cos y^2 \neq y\cos y^2 = \frac{\partial N}{\partial x}.$$

Hence the differential equation is not exact. Now

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3}{x},$$

Therefore the integrating factor is

$$e^{\int \frac{3}{x}dx} = x^3.$$

Multiplying the given equation by x^3, we get the new equation

$$2x^3\sin y^2 dx + x^4 y\cos y^2 dy = 0.$$

This equation is exact because

$$\frac{\partial}{\partial y}(2x^3\sin y^2) = 4x^3 y\cos y^2 = \frac{\partial}{\partial x}(x^4 y\cos y^2).$$

Thus Solution of the given differential equation is

$$\frac{x^4\sin y^2}{2} = c = \text{constant}.$$

Substituting the initial condition $y(2) = \sqrt{\frac{\pi}{2}}$ in to the solution we have,

$$c = 8.$$

Hence the desired particular solution is

$$x^4\sin y^2 = 16.$$

Exercise: Find an integrating factor and solve the following differential equation

1. $2xydx + 3x^2dy = 0.$
2. $(2\cos y + 4x^2)dx = x\sin ydy.$
3. $x^{-1}\cosh ydx + \sinh ydy = 0.$
4. $(xy - 1)dx + (x^2 - xy)dy = 0.$
5. $(\sin x + \cos x\tan y)(dx + dy) + 2\sin ydy = 0.$

9 Linear Differential equation, Bernoulli Equation

The differential equation of the form

$$y' + p(x)y = r(x)$$

is called linear differential equation equation where as equation of the form

$$y' + p(x)y = r(x)y^n$$

is called Bernoulli Equation.

How to solve
Given equation of the form $y' + p(x)y = r(x)$.
Integrating factor $\mu = e^{\int P(x)dx}$.
Solution $y = \frac{1}{\mu}\{\int r(x)\mu dx.\}$

Example 9.1.
$y' - y = e^{2x}$.

Solution 9.2. Here $p = -1, r = e^{2x}, \mu = e^{\int(-1)dx} = e^{-x}$.
$y(x) = e^{x}\left[\int e^{-x}e^{2x}dx + c\right] = ce^{x} + e^{2x}$

Solution of Bernoulli Equation

Example 9.3.
$y' - ay = -by^2$

Solution 9.4. Dividing y^2 throughout the equation and taking $u = y^{-1}$ we get the resulting equation

$$u' + au = b$$

which is a linear equation, taking $p = a, r = b$, we get $\mu = e^{ax}$
The solution $u = e^{-ax}\left[\frac{b}{a}e^{ax} + c\right] = ce^{-ax} + \frac{b}{a}$.
$y = \frac{1}{u} = \frac{1}{(\frac{b}{a}) + ce^{-ax}}$

Home work
Solve the differential equation

1. $y' + 4y = \cos x$.

2. $x^2 y' + 2xy = \sin h5x$.

3. $y' = (y-2)\cot x$.

4. $y' + xy = xy^{-1}$.

5. $y' = \frac{1}{6e^y - 2x}$

10 Linear Differential Equations of second and higher order

A second order differential equation is called linear if it can be written

$$y'' + p(x)y' + q(x)y = r(x)$$

and nonlinear if it cannot be written in this form. If $r(x) = 0$ in the above equation, then the differential equation is called homogeneous, otherwise non-homogeneous.

Theorem 10.1. *For a homogeneous linear second order differential equation, any linear combination of two solutions on an open interval I is again a solution of that equation on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.*

Proof. Let y_1 and y_2 be two solutions. The by substituting $y = c_1y_1 + c_2y_2$ and its derivative into the homogeneous second order differential equation, we get

$$\begin{aligned} y'' + py' + qy &= (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2) \\ &= c_1y_1'' + c_2y_2'' + p(c_1y_1' + c_2y_2') + q(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + py_1' + qy) + c_2(y_2'' + py_2' + qy_2) \\ &= 0 \end{aligned}$$

This proves that y is a solution of the second order differential equation on I. $\square$

11 Second order Homogeneous Equations with constant coefficients

Consider the second order homogeneous linear differential equation

$$y'' + ay' + by = 0 \tag{7}$$

whose coefficients a, b are constants. To Find the solution of the equation (7), we put $y = e^{\lambda x}, y' = \lambda e^{\lambda x}, y'' = \lambda^2 e^{\lambda x}$ into the equation (7), we obtain

$$\lambda^2 + a\lambda + b = 0. \tag{8}$$

This equation is called the **characteristic equation** (or auxiliary equation) of (7). If the roots of the equation (8) are **real and distinct** (say λ_1, λ_2) then the solution of the equation (7) is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

If the roots of equation (8) are repeated real roots (say $\lambda_1 = \lambda_2$) then solution of equation (7) is

$$y = (c_1 + c_2 x)e^{\lambda_1 x}.$$

In case of complex roots (say $\alpha \pm i\beta$) of equation (8) then the solution is

$$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$$

where A, B are constants.

Example 11.1.
Solve $y'' - y' - 6 = 0$.

Solution 11.2. The auxiliary equation is $\lambda^2 - \lambda - 6 = 0$. That implies $\lambda = 3, -2$ which are real and distinct. Hence the general solution is

$$y = c_1 e^{3x} + c_2 e^{-2x}.$$

Example 11.3.
Solve $y'' + 8y' + 16y = 0$.

Solution 11.4. The characteristic equation is $\lambda^2 + 8\lambda + 16 = 0$. It has the double root $\lambda = -4$. Hence the general solution is

$$y = (c_1 + c_2 x)e^{-4x}.$$

Example 11.5.
Solve $y'' + 4y' + 5y = 0$.

Solution 11.6. The auxiliary equation is $\lambda^2 + 4\lambda + 5 = 0$. Solving for λ, we get $\lambda = -2 \pm i$. Hence the general solution is

$$y = e^{-2x}(A\cos x + B\sin x).$$

Home work

1. $10y'' + 6y' - 4y = 0$.
2. $9y'' - 30y' + 25y = 0$.
3. $y'' + 4y' + 4y = 0$.
4. $y'' - 5y' + 6y = 0$.
5. $y'' + 0.2y' + 4.01y = 0$.

12 Euler-Cauchy Equation

Consider the equation

$$x^2y'' + axy' + by = 0, \qquad a, b \text{ constants} \tag{9}$$

which is called the Euler-Cauchy Equation. **Working rule for finding the Euler-Cauchy equation:**

Put

$$y = x^m, \ y' = mx^{m-1}, \ y'' = m(m-1)x^{m-2}$$

into the (9) and equating the coefficients of x^m to zero we obtain

$$m^2 + (a-1)m + b = 0$$

which as called the auxiliary equation. **Three cases of solution:**

case1 If the auxiliary equation has distinct real roots $m_1, \ m_2$ the corresponding general solution of (9) is

$$y = c_1x^{m_1} + c_2x^{m_2}.$$

case2 If the auxiliary equation has double root say $m_1 = m_2$ the the corresponding general solution is

$$y = (c_1 + c_2\ln x)x^{m_1}.$$

case3 If the auxiliary equation has complex conjugate roots, say, $m_1 = \mu + i\nu,\ m_2 = \mu - i\nu$ then the corresponding general solution of (9) is

$$y = x^{\mu}[A\cos(\nu \ln x) + B\sin(\nu \ln x)].$$

Example 12.1.
Solve the Euler-Cauchy Equation

$$x^2y'' - 2.5xy' - 2.0y = 0$$

Solution 12.2. The auxiliary equation is $m^2 - 3.1m - 2 = 0$. The roots are $m_1 = -0.5,\ \ m_2 = 4$. So the general solution is $y = c_1x^{-0.5} + c_2x^4$.

Example 12.3.
Solve the Euler-Cauchy Equation

$$x^2y'' - 3xy' + 4y = 0$$

Solution 12.4. The auxiliary equation has the double root $m = 2$. So the general solution is $y = (c_1 + c_2 \ln x)x^2$.

Example 12.5.
Solve the Euler-Cauchy Equation

$$x^2y'' + 7xy' + 13y = 0$$

Solution 12.6. The auxiliary equation is $m^2 + 6m + 13 = 0$. The roots are $m_1 = -3 \pm 2i$ So the general equation is $y = x^{-3}(A\cos 2\ln x + B\sin 2\ln x)$.

Home work Solve

1. $x^2y'' + xy' + y = 0$.
2. $10x^2y'' + 46xy' + 32.4y = 0$.
3. $x^2y'' - 20y = 0$.
4. $4x^2y'' + 24xy' + 25y = 0$.
5. $x^2y'' - 2xy' + 2y = 0$.

13 Solution by undetermined coefficients

Consider the differential equation

$$y'' + ay' + by = r(x) \tag{10}$$

Table 1: Method of undetermined coefficients.

Term in $r(x)$	Choice for y_p
$ke^{\gamma(x)}$	$ce^{\gamma x}$
$kx^n, \;\; n = 0, 1, 2, 3, \cdots$	$k_n x^n + k_{n-1}x^{n-1} + \cdots k_1 x + k_0$
$k\cos\omega x$	$k\cos\omega x + m\sin\omega x$
$k\sin\omega x$	$k\cos\omega x + m\sin\omega x$
$ke^{\alpha(x)}\cos\omega x$	$e^{ax}(k\cos\omega x + m\sin\omega x)$
$ke^{\alpha(x)}\sin\omega x$	$e^{ax}(k\cos\omega x + m\sin\omega x)$

A general solution of a nonhomogeneous linear differential equation (10) is a sum of of the form

$$y = y_h + y_p$$

where y_h is a general solution of the corresponding homogeneous equation and y_p is any particular solution of nonhomogeneous equation. Here our main task is to discuss methods for finding such y_p.

Rules for the method od undetermined coefficients

(A) Basic Rule: If ($r(x)$ in (10) is one of the functions in the first column in the Table1, chose the corresponding function y_p in the second column and determined its undetermined coefficients by substituting y_p and its derivatives into (10).

Modification Rule:

If a term in your choice for for y_p happens to be a solution of the homogenous equation corresponding to (10), then multiply your choice of y_p by x (or by x^2 of this solution corresponds to a double root of the characteristic equation of the homogeneous equation).

Sum rule

If $r(x)$ is a sum functions in several lines of table 1, first column then choose for y_p the sum of the functions in the corresponding lines of the second column.

Example 13.1.

Solve the nonhomogeneous equation

$$y^{''} + 4y = 8x^2.$$

Solution 13.2. By using the table (1), we choose

$$y_p = K_2x^2 + K_1x + K_0.$$

Then $y_p^{''} = 2K_2$. Substitution gives

$$2K_2 + 4(K_2x^2 + K_1x + K_0) = 8x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have $4K_2 = 8, 4K_1 = 0, 2K_2 + 4K_0 = 0$. Thus $K_2 = 2, K_1 = 0, K_0 = -1$. Hence $y_p = 2x^2 - 1$, and a general solution of the equation is $y = y_h + y_p = A\cos 2x + B\sin 2x + 2x^2 - 1$.

Exercise

Find a general solution of the given differential equation using method of undetermined coefficients.

1. $y^{''} + 10y' + 25y = e^{-5x}$.
2. $y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$.
3. $y'' - y' - \frac{3}{4}y = 21\sin h2x$.
4. $y'' + y' + 9.25y = 9.25(4 + e^{-x})$.
5. $y'' + 1.2y' + 0.36y = 4e^{-0.6x}$.
6. $y'' - 3y' = 28\cos h4x$.
7. $3y'' + 10y' + 3y = \sec x$.
8. $y'' - 12y' + y = -6x^3 + 3x^2 + 377\sin x$.

14 Solution by variation of parameters

Consider the differential equation

$$y'' + p(x)y' + q(x)y = r(x) \tag{11}$$

with arbitrary variable function p, q, and r that are continuous on some interval I. This method gives a particular solution y_p of (11) on I in the form

$$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

where y_1, y_2 form a basis of solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

corresponding to (11) and

$$W = y_1 y_2' - y_2 y_1'$$

is the wronskian of y_1, y_2.

Example 14.1.
Find the particular integral of

$$y'' + y = cosec\ x$$

using method of variation of parameters.

Solution 14.2. The complementary function is

$$y_h = c_1 \cos x + c_2 \sin x.$$

Here $y_1 = \cos x, y_2 = \sin x$. The Wronskian $W = 1$. Hence $y_p = -x\cos x + \sin x \log(\sin x)$.

Home Work

Solve the following differential equations using method of variation of parameters.

1. $y'' + y = \sec x$.
2. $y'' + 4y' + 4y = \frac{2e^{-2x}}{x^2}$.
3. $y'' - 2y' - 3y = 2e^x - 10\sin x$.
4. $y'' + y = x\sin x$.
5. $y'' + 2y' + 4y = \cos 4x$.
6. $y'' + 9y = cosec\ x$

UNIT-IV

15 Series solution of Differential Equations

To solve the homogeneous differential equations with constant coefficients we have used some algebraic methods. But to find or solve the differential equation with variable coefficient such as Legender's equations, Bessel's equation we need to use power series method.
i.e., Let

$$y'' + p(x)y' + q(x)y = 0,$$

we assume here a solution in the form of a power series with unknown coefficient as,

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

is a solution of a given differential equation.
Differentiating with respect to x

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots .$$

Substituting $y,\ y',\ y''$ in the given differential equation of the second order, we have the power series form of differential equation.

To find the solution of differential equation, we have to get the values of $a_0,\ a_1,\ a_2, \cdots$. To obtain these coefficients, we have to equate with the power of x. After finding the values and putting in the given equation, we get the solution of the given differential equation.

Example 15.1.
Solve

$$y' = 3y. \tag{12}$$

Solution 15.2. Let

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

is a solution of a given differential equation.
Differentiating with respect to x, we have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots .$$

Substituting this in (12), we have

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = 3a_0 + 3a_1 x + 3a_2 x^2 + \cdots .$$

By comparing the coefficients of same power of x from above equation, we have

$$a_1 = 3a_0,$$

$$2a_2 = 3a_1 = 3(3a_0) = 9a_0,$$
$$\Rightarrow a_2 = \frac{9}{2}a_0.$$

Again

$$3a_2 = 3a_3,$$
$$\Rightarrow a_3 = a_2 = \frac{9}{2}a_0.$$

Substituting these values in the solution, we have

$$y = a_0 + 3a_0 x + \frac{9}{2}a_0 x^2 + \cdots .$$

$$\Rightarrow y = a_0(1 + 3x + \frac{9}{2}x^2 + \cdots).$$

Example 15.3.
Find the power series solution of

$$(1 - x)y' = y. \tag{13}$$

Solution 15.4. Let

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

is a solution of a given differential equation.
Differentiating with respect to x, we have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots .$$

Substituting this in (13), we have

$$(1 - x)(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\Rightarrow a_1 + 2a_2x + 3a_3x^2 + \cdots - a_1x - 2a_2x^2 - 3a_3x^3 - \cdots = a_0 + a_1x + a_2x^2 + \cdots .$$

$$\Rightarrow a_1 + 2a_2x + 3a_3x^2 + \cdots = a_0 + 2a_1x + 3a_2x^2 + \cdots .$$

comparing the coefficients of x, we have

$$a_0 = a_1,$$

$$2a_2 = 2a_1 = a_0,$$

$$\Rightarrow a_2 = a_1 = a_0.$$

$$3a_3 = 3a_2,$$

$$\Rightarrow a_3 = a_2 = a_1 = a_0.$$

Substituting these values in the solution, we have

$$y = a_0 + a_0x + a_0x^2 + \cdots .$$

$$\Rightarrow y = a_0(1 + x + x^2 + \cdots).$$

Home Work
Solve the following differential equations using power series method.

1. $y' = 3x^2y$.
2. $y'' = y$.
3. $y' + 2y = 0$.
4. $xy' + 3y = 0$.

16 Theory of Power-series Method; Radius of Convergence

Consider the power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots , \tag{14}$$

where $a_0,\ a_1,\ a_2, \cdots$ are the constants called the coefficients of the power series. x_0 is a constant called the center of the series and x is a variable.

Some Expansion of Functions:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n. \qquad (|x| < 1)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Let us take the n^{th} partial sum of (14),

$$S_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n.$$

If for some $x = x_1$.

$$\lim_{n\to\infty} S_n(x_1) = S(x_1).$$

If $S(x_1)$ is finite, then the series convergent at $x = x_1$, otherwise divergent.

17 Radius of Convergence or Circle of Convergence:

A circle $|x - x_0| = R$ for which the power series of the form $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is convergent is known as circle of convergence and R is known as the radius of convergence.

How to find the radius Of convergence of a power series:

The radius of convergence R of the power series can be calculated by

$$\frac{1}{R} = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$$

or

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}.$$

Example 17.1.
Find the radius of convergence of the power series

$$\sum_{m=0}^{\infty} \frac{x^{2m}}{m!}.$$

Solution 17.2. Compare with the power series of the form $\sum_{n=0}^{\infty} a_n x^n$.
Put $x^2 = z$, Now the series becomes $\sum_{m=0}^{\infty} a_m x^m$.
Here $a_m = \frac{1}{m!}$, and $a_{m+1} = \frac{1}{(m+1)!}$ So

$$|\frac{a_{m+1}}{a_m}| = \frac{1}{(m+1)!}.m! = \frac{1}{(m+1)}.$$

Now

$$\frac{1}{R} = \lim_{m\to\infty} \frac{1}{(m+1)} = 0.$$

$$\Rightarrow R = \infty.$$

$$\Rightarrow \mathit{Radius\ \ of\ \ convergence\ \ \ is\ \ } \infty.$$

Example 17.3.
Find the radius of convergence

$$\sum_{n=1}^{\infty} (n+1)nx^n,$$

and

$$\sum_{n=0}^{\infty} \frac{n!}{n^n}(z+\pi)^n.$$

Solution 17.4. In the first problem $a_n = (n+1)n$ and $a_{n+1} = (n+2)(n+1)$.

$$\frac{1}{R} = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(n+2)(n+1)}{(n+1)n}| = \lim_{n\to\infty} |\frac{(n+2)}{n}| = 1.$$

$$\Rightarrow R = 1.$$

In the second problem $a_n = \frac{n!}{n^n}$, and $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$.
Now

$$\frac{1}{R} = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!}| = \lim_{n\to\infty} |\frac{1}{(1+\frac{1}{n})^n}| = \frac{1}{e}.$$

$$\Rightarrow R = e.$$

Definition 17.5. A real function $f(x)$ is called analytic at a point $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

Home Work

Determine the radius of convergence of the following series.

1. $\sum_{m=0}^{\infty}(-1)^m x^{4m}$.

2. $\sum_{m=0}^{\infty} m^m x^m$.

3. $\sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$.

4. $\sum_{m=0}^{\infty} \frac{(x-x_0)^{2m}}{2^m}$.

5. $\sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m x^{2m}$.

18 Legendre's Equation, Legendre Polynomials

$P_n(x)$:

Consider

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{15}$$

where n is a real number and the Solution of (15) is Legendre function.
Dividing the coefficient of y'', we have

$$y'' - \frac{2x}{(1-x^2)}y' + \frac{n(n+1)}{(1-x^2)}y = 0.$$

Now applying power series method; we have

$$y = \sum_{m=0}^{\infty} a_m x^m,$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1},$$

and

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}.$$

Substituting the values in (15), we have

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0.$$

By writing the first expression as two separate series, we have

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0.$$

By writing out each series and arranging each power in a column, we obtain,

$$2a_2 + 6a_3x + 12a_4x^2 + \cdots + (s+2)(s+1)a_{s+2}x^s \cdots$$

$$-2a_2x^2 - \cdots + \cdots$$

$$-2a_1x - 4a_2x^2 - \cdots - s(s-1)a_sx^s - \cdots$$

$$n(n+1)a_0 + n(n+1)a_1x + n(n+1)a_2x^2 + \cdots - 2a_sx^s - \cdots = 0.$$

Equating the coefficients of each power of x to zero, we obtain,

$$n(n+1)a_0 + 2a_2 = 0.$$

$$[n(n+1)-2]a_1 + 6a_3 = 0.$$

$$[n(n+1)-5]a_2 + 12a_4 = 0.$$

and in generally for $s = 2, 3, 4, \cdots,$

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s \qquad (s = 0, 1, 2, \cdots)$$

This is called a recurrence relation. By inserting these values, we obtain

$$y(x) = a_0y_1(x) + a_1y_2(x)$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots$$

and

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^2 + \cdots$$

The solution of Legendre's differential equation is called the Legendre Polynomial of degree n and is denoted by $P_n(x)$. In general

$$P_n(x) = \sum_{m=0}^{M}(-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$ whichever is an integer. The First few of these functions are

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

19 Frobenious Method:

Consider

$$y'' + P(x)y' + Q(x)y = 0 \tag{16}$$

This method will applied only for regular singular point if (16).

Singular Point:

A point $x = x_0$ will be a singular point of (16), if $P(x),\ Q(x)$ are not analytic at $x = x_0$.

Regular Singular Point:

Let $x = x_0$ be a singular point of (16). Then the singular point is said to be regular singular point if $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$.

Example 19.1.

$$(x-1)^2 y'' + 2x(x-1)y' + 3(x+1)y = 0$$
$$\Rightarrow y'' + \frac{2x(x-1)}{(x-1)^2} y' + \frac{3(x+1)}{(x-1)^2} y = 0.$$
$$Here\ \ P(x) = \frac{2x(x-1)}{(x-1)^2}, \quad Q(x) = \frac{3(x+1)}{(x-1)^2}.$$

Clearly at $x = 1$, $P(x)$ *and* $Q(x)$ are not analytic. Now

$$(x - x_0)P(x) = (x-1)\frac{2x(x-1)}{(x-1)^2} = 2x,$$

and

$$(x - x_0)^2 Q(x) = (x-1)^2 \frac{3(x+1)}{(x-1)^2} = 3(x+1).$$

But both are analytic at $x = 1$. Now let $x = x_0$ be a regular singular point of (16), then the series solution of (16) can be assumed as $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+m}$, where m is real number satisfying the indicial equation which is of the type

$$m(m-1) + p_0 m + q_0 = 0$$

where $p_0,\ q_0$ are constant terms of $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ respectively.

Case-I

Let $m_1,\ m_2$ be two distinct roots of the indicial equation, then the two Linearly Independent solutions of (16) is given by

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+m_1}$$

$$y_2(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+m_2}.$$

Case-II

Let $m_1 = m_2 = m$ be double roots of the indicial equation, then a basis is

$$y_1(x) = x^m (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$y_2(x) = y_1(x) \ln x + x^m (a_1 x + a_2 x^2 + \cdots)$$

Case-II

Roots differ by an integer

A basis is

$$y_1(x) = x^{m_1} (a_0 + a_1 x + a_2 x^2 + \cdots)$$

and

$$y_2(x) = k y_1(x) \ln x + x^{m_2} (a_0 + a_1 x + a_2 x^2 + \cdots)$$

where the roots are so denoted that $m_1 - m_2 > 0$ and k may turn out to be zero.

Home work

Find a basis of solutions of the following differential equations.

1. $xy'' + 2y' + 4xy = 0.$
2. $(x+1)^2 y'' + (x+1)y' - y = 0.$
3. $2x(x-1)y'' - (x+1)y' + y = 0.$
4. $x^2 y'' + x^3 y' + (x^2 - 2)y = 0.$

20 Bessel's Equation, Bessel's functions

Consider the Bessel's differential equation

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$$

where ν ia any real non-negative number.Bessel's equation can be solved by using Frobenius method. Substituting $y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}, y', y''$ into the Bessel's equation and equating the coefficient of least power of x to zero, we obtain the indicial equation

$$(r + \nu)(r - \nu) = 0.$$

Thus the solution of the Bessel's differential equation is

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}.$$

This is called the Bessel function of the first kind of order n.

Gamma Function

The gamma function $\Gamma(\nu)$ defined by the integral

$$\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt \qquad (\nu > 0).$$

This yields a basic relationship

$$\Gamma(\nu+1) = \nu\Gamma(\nu).$$

In general

$$\Gamma(n+1) = n! \qquad n = 0, 1, 2, \cdots$$

We list here some important recurrence relationships of Bessel's functions:

$$\frac{d}{dx}[x^{\nu} J_{\nu}(x)] = x^{\nu} J_{\nu-1}(x)$$

$$\frac{d}{dx}[x^{-\nu} J_{\nu}(x)] = -x^{-\nu} J_{\nu+1}(x)$$

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x).$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_{\nu}(x).$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

$$J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

References

[1] Erwin Kreyszig, *Advanced Engineering Mathematics*, John Wiley and Sons, Inc, New York, 2006.

[2] J. Sinha Roy and S.Padhy, *A course on Ordinary and partial differential equations*, Kalyani Publishers, 2010.