

# NETWORK ANALYSIS & SYNTHESIS

## SYLABUS

### **Module-I**

Transients: DC and AC analysis of RL, RC and RLC series circuits. Resonance: Series and Parallel resonance. Loop and node variable analysis, Waveform Synthesis-The Shifted Unit Step, Ramp and Impulse Function, Waveform Synthesis, The Initial and Final Value Theorems, The Convolution Integral.

### **Module-II**

IMPEDANCE FUNCTIONS AND NETWORK THEOREMS: The Concept of Complex Frequency, Transform Impedance and Transform Circuit, Series and parallel Combination of Elements, Superposition and Reciprocity, Thevenin's Theorem and Norton's Theorem.

### **Module-III**

NETWORK FUNCTIONS: POLES AND ZEROS: Terminal Pairs and Ports, Network Function for the One Port and Two Port, The Calculation of Network Function - (a) Ladder Network (b) General Networks. Poles and Zero of Network Functions, Restrictions on Pole and Zero Locations for Driving-Point Functions, Restrictions on Pole and Zero Locations for Transfer Functions, Time-domain Behavior from the Pole and Zero Plot, Stability of Networks.

### **Module-IV**

TWO-PORT PARAMETERS: Relationship of Two-Port Variables, Short-Circuit Admittance parameters, The Open-circuit Impedance Parameters, Transmission parameters, The Hybrid parameters, Relationships Between parameter Sets, Parallel Connection of Two-Port Networks.

### **PART-B**

### **Module-V**

POSITIVE REAL FUNCTION: Driving-Point Functions, Brune's Positive Real Functions, Properties of Positive Real Functions. TESTING DRIVING-POINT FUNCTIONS: An application of the Maximum Modulus Theorem, Properties of Hurwitz Polynomials, The Computation of Residues, Even and Odd functions, Sturm's Theorem, An alternative Test for Positive real functions.

### **Module-VI**

DRIVING-POINT SYNTHESIS WITH LC ELEMENTS: Elementary Synthesis Operations, LC Network Synthesis, RC and RL networks. Properties of RC Network Function, Foster Form of RC Networks, Foster Form of RL Networks, The Cauer Form of RC and RL Networks. RLC ONE TERMINAL - PAIRS: Minimum Positive Real Functions, Brune's Method of RLC Synthesis.

### **Module-VII**

TWO TERMINAL-PAIR SYNTHESIS BY LADDER DEVELOPMENT: Some properties of  $-y$  and  $z$  The LC Ladder Development, Other Considerations, The RC Ladder Development.

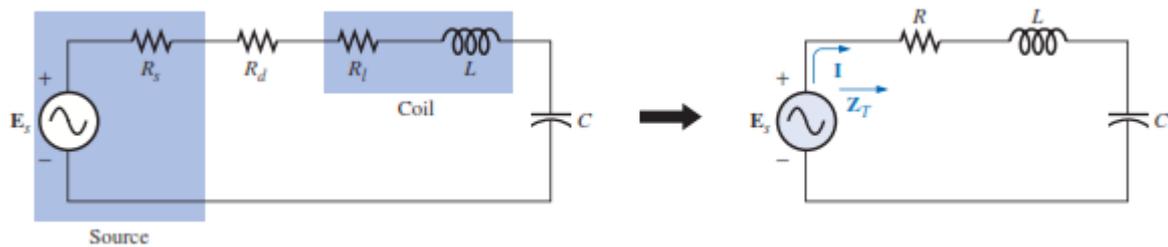
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## MODULE-I

### Resonance: Series and Parallel resonance

#### SERIES RESONANT CIRCUIT

A resonant circuit (series or parallel) must have an inductive and a capacitive element. A resistive element will always be present due to the internal resistance of the source ( $R_s$ ), the internal resistance of the inductor ( $R_l$ ), and any added resistance to control the shape of the response curve ( $R_{design}$ ). The basic configuration for the series resonant circuit appears in Fig with the resistive elements listed above.



The total impedance of this network at any frequency is determined by

$$\mathbf{Z}_T = R + jX_L - jX_C = R + j(X_L - X_C)$$

The resonant conditions described in the introduction will occur when

$$X_L = X_C$$

Removing the reactive component from the total impedance equation. The total impedance at resonance is then simply

$$\mathbf{Z}_{Ts} = R$$

Representing the minimum value of  $\mathbf{Z}_T$  at any frequency. The subscript  $s$  will be employed to indicate series resonant conditions. The resonant frequency can be determined in terms of the inductance and capacitance by examining the defining equation for resonance-

$$X_L = X_C$$

$$f = \frac{1}{2\pi\sqrt{LC}}$$

which you will note is the maximum current for the circuit of Fig for an applied voltage  $\mathbf{E}$  since  $\mathbf{Z}$  is a minimum value. Consider also that *the input voltage and current are in phase at resonance.*

#### QUALITY FACTOR:

The **quality factor**  $Q$  of a series resonant circuit is defined as the ratio of the reactive power of either the inductor or the capacitor to the average power of the resistor at resonance; that is

$$Q = \frac{\text{reactive power}}{\text{average power}}$$

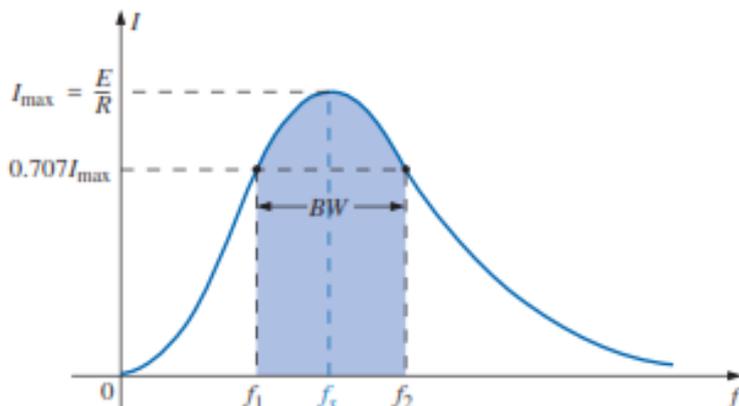
The quality factor is also an indication of how much energy is placed in storage (continual transfer from one reactive element to the other) compared to that dissipated. The lower the level of dissipation for the same reactive power, the larger the  $Q$ , factor and the more concentrated and intense the region of resonance. Substituting for an inductive reactance in Eq. (20.8) at resonance gives us

$$Q = \frac{XL}{R} = \frac{\omega L}{R}$$

Since the quality factor of a coil is typically the information provided by manufacturers of inductors, it is often given the symbol  $Q$  without an associated subscript. It would appear that  $Q$  will increase linearly with frequency since  $X_L = 2\pi fL$ . That is, if the frequency doubles, then  $Q_i$  will also increase by a factor of 2. This is approximately true for the low range to the mid-range of frequencies. Unfortunately, however, as the frequency increases, the effective resistance of the coil will also increase, due primarily to skin effect phenomena, and the resulting  $Q$  will decrease. In addition, the capacitive effects between the windings will increase, further reducing the  $Q_i$  of the coil. For this reason,  $Q$  must be specified for a particular frequency or frequency range. For wide frequency applications, a plot of  $Q$  versus frequency is often provided.

## SELECTIVITY

plot the magnitude of the current  $I = E/Z_T$  versus frequency for a *fixed* applied voltage  $E$ , we obtain the curve shown in Fig, which rises from zero to a maximum value of  $E/R$  (where  $Z_T$  is a minimum) and then drops toward zero (as  $Z$  increases) at a slower rate than it rose to its peak value. The curve is actually the inverse of the impedance-versus-frequency curve. Since the  $Z_T$  curve is not absolutely symmetrical about the resonant frequency, the curve of the current versus frequency has the same property.



There is a definite range of frequencies at which the current is near its maximum value and the impedance is at a minimum. Those frequencies corresponding to 0.707 of the maximum current are called the **band frequencies, cutoff frequencies, or half-power frequencies**. They are indicated by  $f_1$  and  $f_2$  in Fig. The range of frequencies between the two is referred to as the **bandwidth** (abbreviated  $BW$ ) of the resonant circuit.

### EXAMPLE

The bandwidth of a series resonant circuit is 400 Hz.

- If the resonant frequency is 4000 Hz, what is the value of  $Q$ ?
- If  $R = 10 \Omega$ , what is the value of  $X$  at resonance?
- Find the inductance  $L$  and capacitance  $C$  of the circuit.

**Solutions:**

$$\text{a. } BW = \frac{f}{Q} \text{ or } Q = \frac{f}{BW} = \frac{4000 \text{ Hz}}{400 \text{ Hz}} = 10$$

$$\text{b. } Q = \frac{XL}{R} \text{ or } XL = QR = 10 * 10 \text{ ohm} = 100 \text{ ohm}$$

$$\text{c. } XL = 2\pi fL \text{ or } L = \frac{XL}{2\pi f} = \frac{100 \text{ ohm}}{2\pi * 4000} = 3.98 \text{ mH}$$

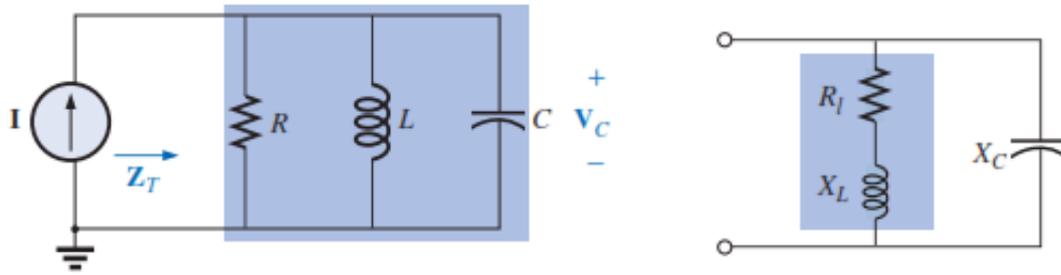
$$XC = \frac{1}{2\pi fC} \text{ or } C = \frac{1}{2\pi fXC} = \frac{1}{2\pi * 4000 * 100 \text{ ohm}} = 0.398 \mu\text{F}$$

### PARALLEL RESONANT CIRCUIT

The basic format of the series resonant circuit is a series  $R-L-C$  combination in series with an applied voltage source. The parallel resonant circuit has the basic configuration of Fig. 20.21, a parallel  $R-L-C$  combination in parallel with an applied current source. For the series circuit, the impedance was a minimum at resonance, producing a significant current that resulted in a high output voltage for  $V_C$  and  $V_L$ . For the parallel resonant circuit, the impedance is relatively high at resonance, producing a significant voltage for  $V_C$  and  $V_L$  through the Ohm's law relationship ( $V_C = IZ_T$ ). For the network of Fig below, resonance will occur when  $X_L = X_C$ , and the resonant frequency will have the same format obtained for series resonance.

If the practical equivalent of Fig below had the format the analysis would be as direct and lucid as that experienced for series resonance. However, in the practical world, the internal resistance of the coil must be placed in series with the inductor. The resistance  $R$  can no longer be included in a simple series or

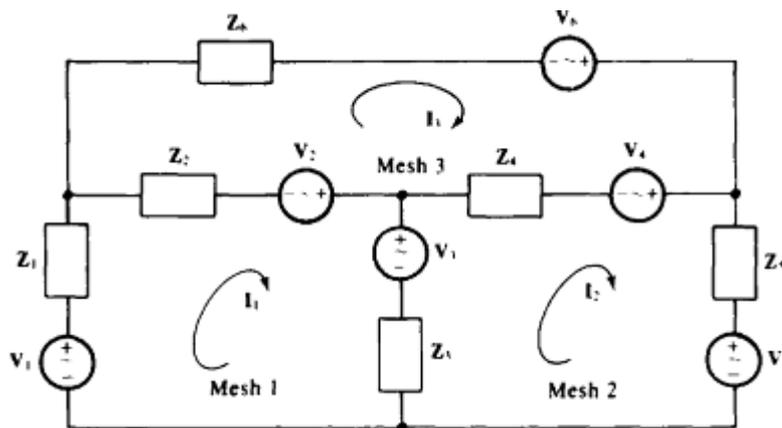
parallel combination with the source resistance and any other resistance added for design purpose.



### Loop and node variable analysis

Mesh analysis for phasor-domain circuits should be apparent from the presentation of mesh analysis for dc circuits. Preferably all current sources are transformed to voltage sources, then clockwise-referenced mesh currents are assigned, and finally **KVL** is applied to each mesh.

$$I_1 Z_1 + (I_1 - I_3) Z_2 + (I_1 - I_2) Z_3 = V_1 + V_2 - V_3$$



where  $I_1 Z_1$ ,  $(I_1 - I_3) Z_2$ , and  $(I_1 - I_2) Z_3$  are the voltage drops across the impedances  $Z_1$ ,  $Z_2$ , and  $Z_3$ . Of course, is the sum of the voltage rises from voltage sources in mesh 1. As a memory aid, a source voltage is added if it “aids” current flow -that is, if the principal current has a direction out of the positive terminal of the source. Otherwise, the source voltage is subtracted. This equation simplifies to

$$(Z_1 + Z_2 + Z_3) I_1 - Z_3 I_2 - Z_2 I_3 = V_1 + V_2 - V_3$$

The coefficient of  $I_1$  is the *self-impedance* of mesh 1, which is the sum of the impedances of mesh 1. The  $-Z_3$  coefficient of  $I_2$  is the negative of the impedance in the branch common to meshes 1 and 2. This impedance  $Z_3$  is a mutual impedance, it is mutual to meshes 1 and 2. Likewise, the  $-Z_2$  coefficient of  $I_3$  is the negative of the impedance in the branch mutual to meshes 1 and 3, and so  $Z_2$ ,

is also a mutual impedance. It is important to remember in mesh analysis that the mutual terms have initial negative signs.

It is, of course, easier to write mesh equations **using** self-impedances and mutual impedances than it is to directly apply KVL. Doing this for meshes 2 and 3 results in

$$-Z_3 I_1 + (Z_3 + Z_4 + Z_5) I_2 - Z_4 I_3 = V_3 + V_4 - V_5$$

and

$$-Z_2 I_1 - Z_4 I_2 + (Z_2 + Z_4 + Z_6) I_3 = -V_2 - V_4 + V_6$$

Placing the equations together shows the symmetry of the **I** coefficients about the principal diagonal

$$\begin{array}{r} (Z_1 + Z_2 + Z_3) I_1 - \quad \quad Z_3 I_2 - \quad \quad Z_2 I_3 = V_1 + V_2 - V_3 \\ \quad -Z_3 I_1 + (Z_3 + Z_4 + Z_5) I_2 - \quad \quad Z_4 I_3 = V_3 + V_4 - V_5 \\ \quad -Z_2 I_1 - \quad \quad Z_4 I_2 + (Z_2 + Z_4 + Z_6) I_3 = -V_2 - V_4 + V_6 \end{array}$$

Usually, there is no such symmetry if the corresponding circuit has dependent sources. **Also**, some of the off-diagonal coefficients may not have initial negative signs. This symmetry of the coefficients is even better seen with the equations written in matrix form:

$$\begin{bmatrix} Z_1 + Z_2 + Z_3 & -Z_3 & -Z_2 \\ -Z_3 & Z_3 + Z_4 + Z_5 & -Z_4 \\ -Z_2 & -Z_4 & Z_2 + Z_4 + Z_6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_1 + V_2 - V_3 \\ V_3 + V_4 - V_5 \\ -V_2 - V_4 + V_6 \end{bmatrix}$$

Loop analysis is similar except that the paths around which KVL is applied are not necessarily meshes, and the loop currents may not all be referenced clockwise. **So**, even if a circuit has no dependent sources, some of the mutual impedance coefficients may not have initial negative signs. Preferably, the loop current paths are selected such that each current source has just one loop current through it. Then, these loop currents become known quantities with the result that it is unnecessary to write KVL equations for the loops or to transform any current sources to voltage sources.

### Node Analysis:

Nodal analysis for phasor-domain circuits is similar to nodal analysis for dc circuits. Preferably, all voltage sources are transformed to current sources. Then, a reference node is selected and all other nodes are referenced positive in potential with respect to this reference node. Finally, KCL is applied to each non reference node. Often the polarity signs for the node voltages are not shown

because of the convention to reference these voltages positive with respect to the reference node.

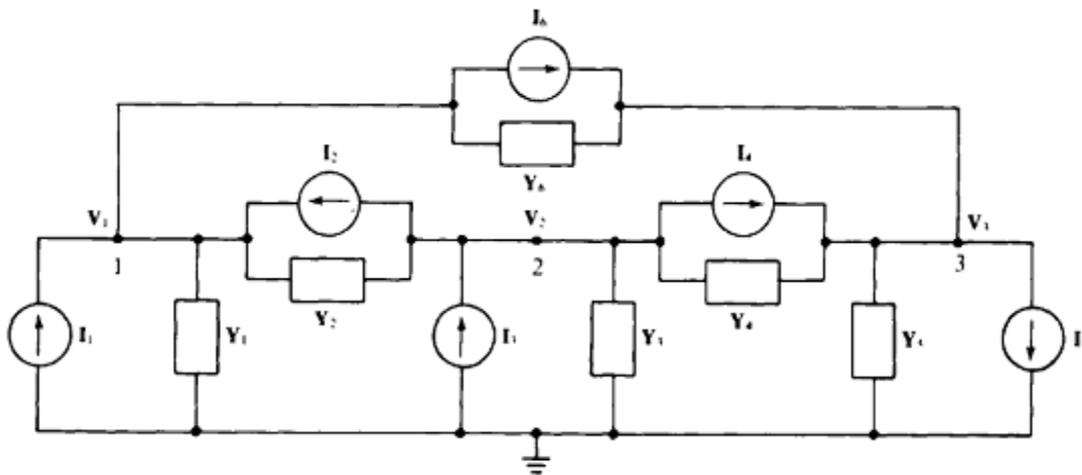
For an illustration

of nodal analysis applied to a phasor-domain circuit, consider the circuit shown.

The KCL equation for node 1 is

$$\mathbf{v}_1\mathbf{Y}_1 + (\mathbf{v}_1 - \mathbf{V}_2)\mathbf{y}_2 + (\mathbf{v}_1 - \mathbf{v}_3)\mathbf{y}_6 = \mathbf{I}_1 + \mathbf{I}_2 - \mathbf{I}_6$$

where  $\mathbf{V}_1\mathbf{Y}_1$ ,  $(\mathbf{V}_1 - \mathbf{V}_2)\mathbf{Y}_2$  and  $(\mathbf{V}_1 - \mathbf{v}_3)\mathbf{Y}_6$  are the currents flowing away from node 1 through the admittances  $\mathbf{Y}_1$ ,  $\mathbf{Y}_2$  and  $\mathbf{Y}_6$ . Of course,  $\mathbf{I}_1 + \mathbf{I}_2 - \mathbf{I}_6$  is the sum of the currents flowing into node 1 from current sources.



This equation simplifies to

$$(\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_6)\mathbf{V}_1 - \mathbf{Y}_2\mathbf{V}_2 - \mathbf{Y}_6\mathbf{V}_3 = \mathbf{I}_1 + \mathbf{I}_2 - \mathbf{I}_6$$

The coefficient  $\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_6$  of  $\mathbf{V}_1$  is the *self-admittance* of node 1, which is the sum of the admittances connected to node 1. The coefficient  $-\mathbf{Y}_2$  of  $\mathbf{V}_2$  is the negative of the admittance connected between nodes 1 and 2. **So**,  $\mathbf{Y}_2$  is a *mutual admittance*. Similarly, the coefficient  $-\mathbf{Y}_6$  of  $\mathbf{V}_3$  is the negative of the admittance connected between nodes 1 and 3, and so  $\mathbf{Y}_6$  is also a mutual admittance.

It is, of course, easier to write nodal equations using self-admittances and mutual admittances than, it is to directly apply KCL. Doing this for nodes 2 and 3 produces

$$-\mathbf{Y}_2\mathbf{V}_1 + (\mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4)\mathbf{V}_2 - \mathbf{Y}_4\mathbf{V}_3 = -\mathbf{I}_2 + \mathbf{I}_3 - \mathbf{I}_4$$

$$-\mathbf{Y}_6\mathbf{V}_1 - \mathbf{Y}_4\mathbf{V}_2 + (\mathbf{Y}_4 + \mathbf{Y}_5 + \mathbf{Y}_6)\mathbf{V}_3 = \mathbf{I}_4 - \mathbf{I}_5 + \mathbf{I}_6$$

Usually, there is no such symmetry if the corresponding circuit has dependent sources. **Also**, some of the off-diagonal coefficients may not have initial -ve signs. In matrix form these equations are

$$\begin{bmatrix} Y_1 + Y_2 + Y_6 & -Y_2 & -Y_6 \\ -Y_2 & Y_2 + Y_3 + Y_4 & -Y_4 \\ -Y_6 & -Y_4 & Y_4 + Y_5 + Y_6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_1 + I_2 - I_6 \\ -I_2 + I_3 - I_4 \\ I_4 - I_5 + I_6 \end{bmatrix}$$

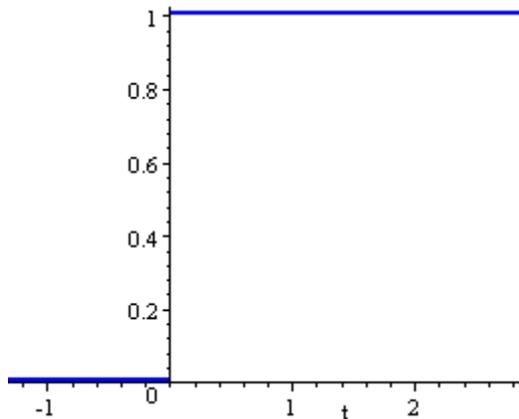
## Waveform Synthesis-The Shifted Unit Step

THE UNIT STEP FUNCTION:

Definition: The unit step function,  $u(t)$ , is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

That is,  $u$  is a function of time  $t$ , and  $u$  has value **zero** when time is negative and value **one** when time is positive



In many circuits, waveforms are applied at specified intervals other than  $t=0$ . Such a function may be described using the **shifted** (aka **delayed**) unit step function.

### Definition of Shifted Unit Step Function

A function which has value 0 up to the time  $t=a$  and thereafter has value 1, is written:

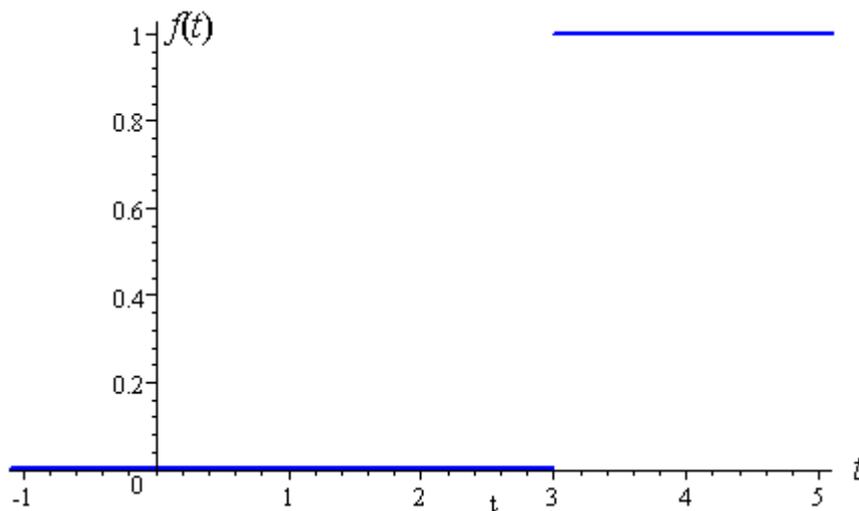
$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

### Example 1 - Shifted Unit Step Function

$$f(t) = u(t-3)$$

The equation means  $f(t)$  has value of 0 when  $t < 3$  and 1 when  $t > 3$ .

The sketch of the waveform is as follows:



### The Initial and Final Value Theorems, The Convolution Integral.

#### INITIAL VALUE THEOREM

The **initial value theorem** is a theorem used to relate frequency domain expressions to the time domain behavior as time approaches zero

Let

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

be the (one-sided) Laplace transform of  $f(t)$ . The initial value theorem then says

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s).$$

Based on the definition of Laplace transform of derivative we have:

$$sF(s) = f(0^-) + \int_{t=0^-}^{\infty} e^{-st} f'(t) dt$$

thus:

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} [f(0^-) + \int_{t=0^-}^{\infty} e^{-st} f'(t) dt]$$

But  $\lim_{s \rightarrow \infty} e^{-st}$  is indeterminate between  $t=0^-$  to  $t=0^+$ ; to avoid this, the integration can be performed in two intervals:

$$\lim_{s \rightarrow \infty} [\int_{t=0^-}^{\infty} e^{-st} f'(t) dt] = \lim_{s \rightarrow \infty} \{ \lim_{\epsilon \rightarrow 0^+} [\int_{t=0^-}^{\epsilon} e^{-st} f'(t) dt] + \lim_{\epsilon \rightarrow 0^+} [\int_{t=\epsilon}^{\infty} e^{-st} f'(t) dt] \}$$

In the first expression where  $0^- < t < 0^+$ ,  $e^{-st} = 1$ . In the second expression, the order of integration and limit-taking can be changed. Also  $\lim_{s \rightarrow \infty} e^{-st}(t)$  where  $0^+ < t < \infty$  is zero. Therefore,

$$\begin{aligned} \lim_{s \rightarrow \infty} \left[ \int_{t=0^-}^{\infty} e^{-st} f'(t) dt \right] &= \lim_{s \rightarrow \infty} \left\{ \lim_{\epsilon \rightarrow 0^+} \left[ \int_{t=0^-}^{\epsilon} f'(t) dt \right] \right\} + \lim_{\epsilon \rightarrow 0^+} \left\{ \int_{t=\epsilon}^{\infty} \lim_{s \rightarrow \infty} [e^{-st} f'(t) dt] \right\} \\ &= f(t) \Big|_{t=0^-}^{t=0^+} + 0 \\ &= f(0^+) - f(0^-) + 0 \end{aligned}$$

By substitution of this result in the main equation we get:

$$\lim_{s \rightarrow \infty} sF(s) = f(0^-) + f(0^+) - f(0^-) = f(0^+)$$

### FINAL VALUE THEOREM

The **final value theorem** (FVT) is one of several similar theorems used to relate frequency domain expressions to the time domain behavior as time approaches infinity. A final value theorem allows the time domain behavior to be directly calculated by taking a limit of a frequency domain expression, as opposed to converting to a time domain expression and taking its limit,

Mathematically, if

$$\lim_{t \rightarrow \infty} f(t)$$

has a finite limit, then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

where  $F(s)$  is the (unilateral) Laplace transform of  $f(t)$ .

Likewise, in discrete time

$$\lim_{k \rightarrow \infty} f[k] = \lim_{z \rightarrow 1} (z - 1)F(z)$$

where  $F(z)$  is the (unilateral) Z-transform of  $f[k]$

By integrating from the definition of Laplace transform of a derivative we have:

$$\lim_{s \rightarrow 0} \int_0^{\infty} \frac{df(t)}{dt} e^{-st} dt = \lim_{s \rightarrow 0} [sF(s) - f(0)]$$

If the infinite integral on LHS exists, then the limit of integral can be written as integral of limit, therefore,

$$\int_0^{\infty} \lim_{s \rightarrow 0} \frac{df(t)}{dt} e^{-st} dt = \int_0^{\infty} df(t) = f(\infty) - f(0)$$

By equating RHSs of previous equations and canceling  $f(0)$  on both sides:

$$f(\infty) = \lim_{s \rightarrow 0} [sF(s)]$$

## CONVOLUTION INTEGRAL

The convolution of  $f$  and  $g$  is written  $f * g$ , using an asterisk or star. It is defined as the integral of the product of the two functions after one is reversed and shifted. As such, it is a particular kind of integral transform:

$$\begin{aligned} (f * g)(t) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} f(t - \tau) g(\tau) d\tau. \quad (\text{commutativity}) \end{aligned}$$

While the symbol  $t$  is used above, it need not represent the time domain. But in that context, the convolution formula can be described as a weighted average of the function  $f(\tau)$  at the moment  $t$  where the weighting is given by  $g(-\tau)$  simply shifted by amount  $t$ . As  $t$  changes, the weighting function emphasizes different parts of the input function.

For functions  $f, g$  supported on only  $[0, \infty)$  (i.e., zero for negative arguments), the integration limits can be truncated, resulting in

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau \quad \text{for } f, g : [0, \infty) \rightarrow \mathbb{R}$$

In this case, the Laplace transform is more appropriate than the Fourier transform below and boundary terms become relevant.

## MODULE II

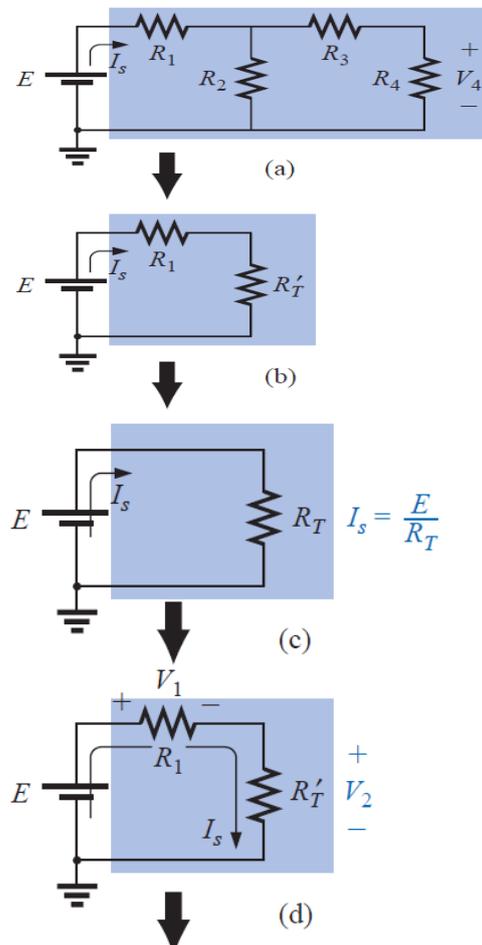
### Series and parallel Combination of Elements

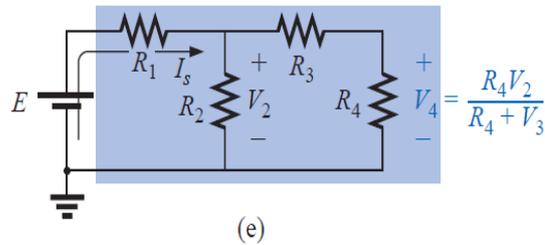
series-parallel networks are networks that contain both series and parallel circuit configurations.

#### General Approach

1. Take a moment to study the problem “in total” and make a brief mental sketch of the overall approach you plan to use. The result may be time- and energy-saving shortcuts.
2. Next examine each region of the network independently before tying them together in series-parallel combinations. This will usually simplify the network and possibly reveal a direct approach toward obtaining one or more desired unknowns. It also eliminates many of the errors that might result due to the lack of a systematic approach.
3. Redraw the network as often as possible with the reduced branches and undisturbed unknown quantities to maintain clarity and provide the reduced networks for the trip back to unknown quantities from the source.
4. When you have a solution, check that it is reasonable by considering the magnitudes of the energy source and the elements in the network. If it does not seem reasonable, either solve the circuit using another approach or check over your work very carefully.

For many single source, series-parallel networks, the analysis is one that works back to the source, determines the source current, and then finds its way to the desired unknown. In Fig (a), for instance, the voltage  $V_4$  is desired. The absence of a single series or parallel path to  $V_4$  from the source immediately reveals that the methods introduced in the last two chapters cannot be applied here. First, series and parallel elements must be combined to establish the reduced circuit of Fig (b). Then series elements are combined to form the simplest of configurations in Fig.(c). The source current can now be determined using Ohm’s law, and we can proceed back through the network as shown in Fig (d). The voltage  $V_2$  can be determined and then the original network can be redrawn, as shown in Fig (e). Since  $V_2$  is now known, the voltage divider rule can be used to find the desired voltage  $V_4$ . Because of the similarities between the networks of Figs (a) and (e), and between (b) and (d), the networks drawn during the reduction phase are often used for the return path.





*Introducing the reduce and return approach.*

The block diagram approach will be employed throughout to emphasize the fact that combinations of elements, not simply single resistive elements, can be in series or parallel. The approach will also reveal the number of seemingly different networks that have the same basic structure and therefore can involve similar analysis techniques.

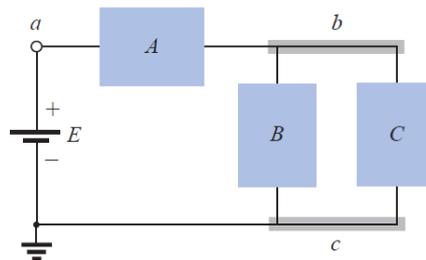
In Fig. blocks *B* and *C* are in parallel (points *b* and *c* in common), and the voltage source *E* is in series with block *A* (point *a* in common). The parallel combination of *B* and *C* is also in series with *A* and the voltage source *E* due to the common points *b* and *c*, respectively.

To ensure that the analysis to follow is as clear and uncluttered as possible, the following notation will be used for series and parallel combinations of elements. For series resistors  $R_1$  and  $R_2$ , a comma will be inserted between their subscript notations, as shown here:

$$R_{1,2} = R_1 + R_2$$

For parallel resistors  $R_1$  and  $R_2$ , the parallel symbol will be inserted between their subscript notations, as follows:

$$R_{1\parallel 2} = R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2}$$



**FIG. 1.2**

*Introducing the block diagram approach.*

**EXAMPLE** If each block of Fig. were a single resistive element, the network of Fig. might result.

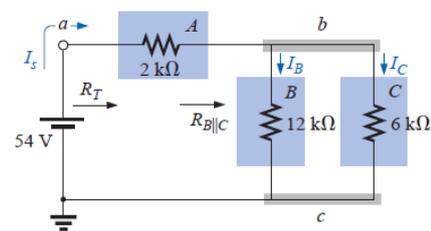
The parallel combination of  $R_B$  and  $R_C$  results in

$$R_{B\parallel C} = R_B \parallel R_C = \frac{(12 \text{ k}\Omega)(6 \text{ k}\Omega)}{12 \text{ k}\Omega + 6 \text{ k}\Omega} = 4 \text{ k}\Omega$$

The equivalent resistance  $R_{B\parallel C}$  is then in series with  $R_A$ , and the total resistance “seen” by the source is

$$R_T = R_A + R_{B\parallel C} = 2 \text{ k}\Omega + 4 \text{ k}\Omega = 6 \text{ k}\Omega$$

The result is an equivalent network, as shown in Fig. permitting the determination of the source current  $I_s$ .



*Example*

$$I_s = \frac{E}{R_T} = \frac{54 \text{ V}}{6 \text{ k}\Omega} = 9 \text{ mA}$$

and, since the source and  $R_A$  are in series,

$$I_A = I_s = 9 \text{ mA}$$

We can then use the equivalent network of Fig. to determine  $I_B$  and  $I_C$  using the current divider rule:

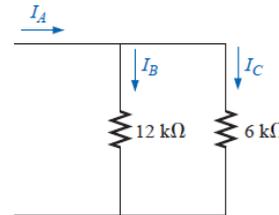
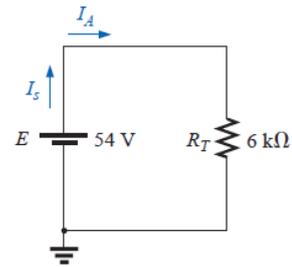
$$I_B = \frac{6 \text{ k}\Omega(I_s)}{6 \text{ k}\Omega + 12 \text{ k}\Omega} = \frac{6}{18} I_s = \frac{1}{3} (9 \text{ mA}) = 3 \text{ mA}$$

$$I_C = \frac{12 \text{ k}\Omega(I_s)}{12 \text{ k}\Omega + 6 \text{ k}\Omega} = \frac{12}{18} I_s = \frac{2}{3} (9 \text{ mA}) = 6 \text{ mA}$$

or, applying Kirchhoff's current law,

$$I_C = I_s - I_B = 9 \text{ mA} - 3 \text{ mA} = 6 \text{ mA}$$

Note that in this solution, we worked back to the source to obtain the source current or total current supplied by the source. The remaining unknowns were then determined by working back through the network to find the other unknowns.



## SUPERPOSITION THEOREM

The superposition theorem states the following:

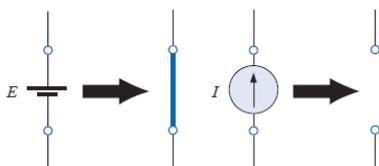
The current through, or voltage across, an element in a linear bilateral network is equal to the algebraic sum of the currents or voltages produced independently by each source.

When one is applying the theorem, it is possible to consider the effects of two sources at the same time and reduce the number of networks that have to be analyzed, but, in general,

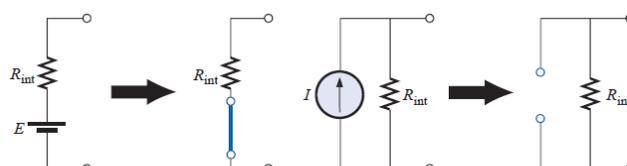
$$\text{Number of networks to be analyzed} = \text{Number of independent sources}$$

To consider the effects of each source independently requires that sources be removed and replaced without affecting the final result. To remove a voltage source when applying this theorem, the difference in potential between the terminals of the voltage source must be set to zero (short circuit); removing a current source requires that its terminals be opened (open circuit). Any internal resistance or conductance associated with the displaced sources is not eliminated but must still be considered.

Figure reviews the various substitutions required when removing an ideal source and Figure reviews the substitutions with practical sources that have an internal resistance.



Removing the effects of ideal sources.

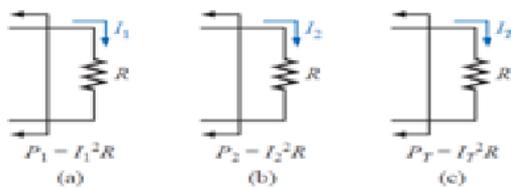


Removing the effects of practical sources.

The total current through any portion of the network is equal to the algebraic sum of the currents produced independently by each source. That is, for a two-source network, if the current produced by one source is in one direction, while that produced by the other is in the opposite direction through the same resistor, *the resulting current is the difference of the two and has the direction of the larger*. If the individual currents are in the same direction, *the resulting current is the sum of two in the direction of either current*. This rule holds true for the voltage across a portion of a network as determined by polarities, and it can be extended to networks with any number of sources. The superposition principle is not applicable to power effects since the power loss in a resistor varies as the square (nonlinear) of the current or voltage.

For instance, the current through the resistor  $R$  of Fig below is  $I_1$  due to one source of a two-source network. The current through the same resistor due to the other source is  $I_2$  as shown in Fig. (b). Applying the superposition theorem, the total current through the resistor due to both sources is  $I_T$ , as shown in Fig. (c)

with  $I_T = I_1 + I_2$



Demonstration of the fact that superposition is not applicable to power effects.

The power delivered to the resistor in Fig. is  $P_1 = I_1^2 R$ , while the power delivered to the same resistor in Fig. (b) is  $P_2 = I_2^2 R$ . If we assume that the total power delivered in Fig.(c) can be obtained by simply adding the power delivered due to each source, we find that

$$P_T = P_1 + P_2 = I_1^2 R + I_2^2 R = I_T^2 R$$

or

$$I_T^2 = I_1^2 + I_2^2$$

This final relationship between current levels is incorrect, however, as can be demonstrated by taking the total current determined by the superposition theorem and squaring it as follows:

$$I_T^2 = (I_1 + I_2)^2 = I_1^2 + I_2^2 + 2I_1 I_2$$

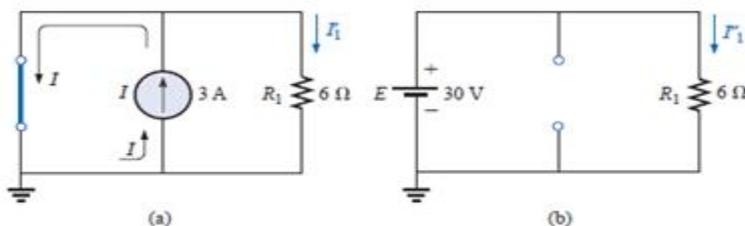
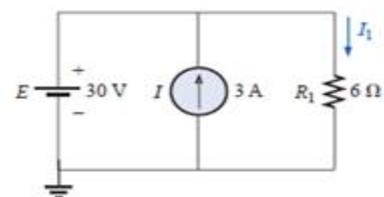
which is certainly different from the expression obtained from the addition of power levels.

In general, therefore,

the total power delivered to a resistive element must be determined using the total current through or the total voltage across the element and cannot be determined by a simple sum of the power levels established by each source.

**EXAMPLE** Determine  $I_1$  for the network of Fig.

**Solution:** Setting  $E = 0$  V for the network of Fig. results in the network of Fig. where a short-circuit equivalent has replaced the 30-V source.



(a) The contribution of  $I$  to  $I_1$ ; (b) the contribution of  $E$  to  $I_1$ .

As shown in Fig. (a), the source current will choose the short-circuit path, and  $I'_1 = 0$  A. If we applied the current divider rule,

$$I'_1 = \frac{R_{sc}I}{R_{sc} + R_1} = \frac{(0 \Omega)I}{0 \Omega + 6 \Omega} = 0 \text{ A}$$

Setting  $I$  to zero amperes will result in the network of Fig. (b), with the current source replaced by an open circuit. Applying Ohm's law,

$$I''_1 = \frac{E}{R_1} = \frac{30 \text{ V}}{6 \Omega} = 5 \text{ A}$$

Since  $I'_1$  and  $I''_1$  have the same defined direction in Fig. (a) and (b), the current  $I_1$  is the sum of the two, and

$$I_1 = I'_1 + I''_1 = 0 \text{ A} + 5 \text{ A} = 5 \text{ A}$$

Note in this case that the current source has no effect on the current through the 6-Ω resistor. The voltage across the resistor must be fixed at 30 V because they are parallel elements.

### RECIPROCALITY THEOREM

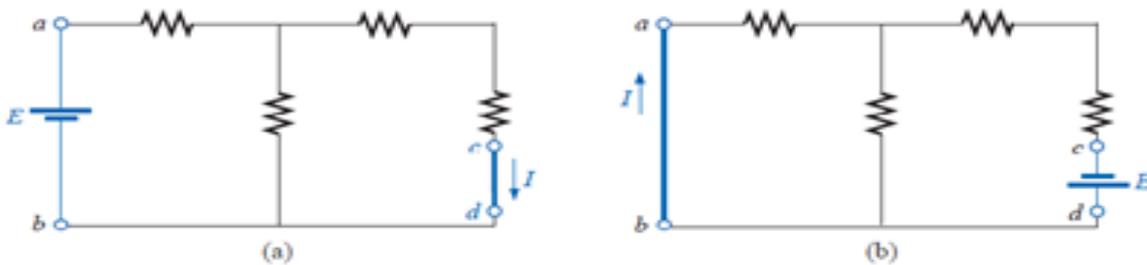
The **reciprocity theorem** is applicable only to single-source networks. It is, therefore, not a theorem employed in the analysis of multisource networks described thus far. The theorem states the following:

The current  $I$  in any branch of a network, due to a single voltage source  $E$  anywhere else in the network, will equal the current through the branch in which the source was originally located if the source is placed in the branch in which the current  $I$  was originally measured.

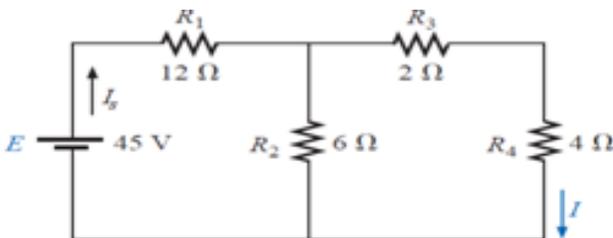
In other words, the location of the voltage source and the resulting current may be interchanged without a change in current. The theorem requires that the polarity of the voltage source have the same correspondence with the direction of the branch current in each position.

In the representative network of Fig., the current  $I$  due to the voltage source  $E$  was determined. If the position of each is interchanged as shown in Fig., the current  $I$  will be the same value as indicated. To demonstrate the validity of this statement and the theorem, consider the network of Fig., in which values for the elements of Fig. have been assigned.

The total resistance is



*Demonstrating the impact of the reciprocity theorem.*



*Finding the current  $I$  due to a source  $E$ .*

$$R_T = R_1 + R_2 \parallel (R_3 + R_4) = 12 \Omega + 6 \Omega \parallel (2 \Omega + 4 \Omega) = 15 \Omega$$

and 
$$I_s = \frac{E}{R_T} = \frac{45 \text{ V}}{15 \Omega} = 3 \text{ A}$$

with 
$$I = \frac{3 \text{ A}}{2} = 1.5 \text{ A}$$

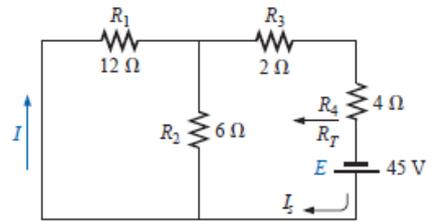
For the network of Fig. which corresponds to that of Fig. we find

$$R_T = R_4 + R_3 + R_1 \parallel R_2$$

$$= 4 \Omega + 2 \Omega + 12 \Omega \parallel 6 \Omega = 10 \Omega$$

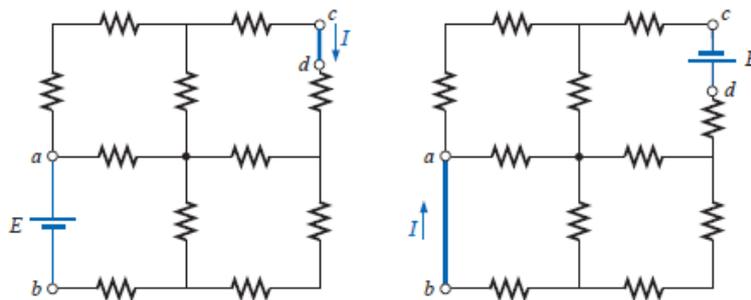
and 
$$I_s = \frac{E}{R_T} = \frac{45 \text{ V}}{10 \Omega} = 4.5 \text{ A}$$

so that 
$$I = \frac{(6 \Omega)(4.5 \text{ A})}{12 \Omega + 6 \Omega} = \frac{4.5 \text{ A}}{3} = 1.5 \text{ A}$$



which agrees with the above.

The uniqueness and power of such a theorem can best be demonstrated by considering a complex, single-source network such as the one shown in Fig.



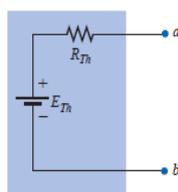
*Demonstrating the power and uniqueness of the reciprocity theorem.*

## THEVENIN'S THEOREM

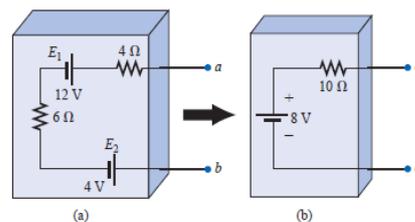
**Thévenin's theorem** states the following:

Any two-terminal, linear bilateral dc network can be replaced by an equivalent circuit consisting of a voltage source and a series resistor, as shown in Fig.

In Fig., for example, the network within the container has only two terminals available to the outside world, labeled *a* and *b*. It is possible using Thévenin's theorem to replace everything in the container with one source and one resistor, as shown in Fig., and maintain the same terminal characteristics at terminals *a* and *b*. That is, any load connected to terminals *a* and *b* will not know whether it is hooked up to the network of Fig. or Fig.. The load will receive the same current, voltage, and power from either configuration of Fig.. Throughout the discussion to follow, however, always keep in mind that the Thévenin equivalent circuit provides an equivalence at the terminals only—the internal construction and characteristics of the original network and the Thévenin equivalent are usually quite different.

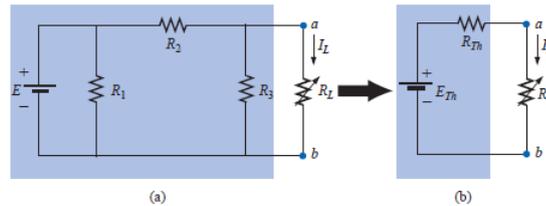


*Thévenin equivalent circuit.*



*The effect of applying Thévenin's theorem.*

In most cases, other elements will be connected to the right of terminals  $a$  and  $b$  in Fig.. To apply the theorem, however, the network to be reduced to the Thévenin equivalent form must be isolated as shown in Fig., and the two “holding” terminals identified. Once the proper Thévenin equivalent circuit has been determined, the voltage, current, or resistance readings between the two “holding” terminals will be the same whether the original or the Thévenin equivalent circuit is connected to the left of terminals  $a$  and  $b$  in Fig.. Any load connected to the right of terminals  $a$  and  $b$  of Fig. will receive the same voltage or current with either network. This theorem achieves two important objectives. First, as was true for all the methods previously described, it allows us to find any particular voltage or current in a linear network with one, two, or any other number of sources. Second, we can concentrate on a specific portion of a network by replacing the remaining network with an equivalent circuit. In Fig., for example, by finding the Thévenin equivalent circuit for the network in the shaded area, we can quickly calculate the change in current through or voltage across the variable resistor  $R_L$  for the various values that it may assume.



Substituting the Thévenin equivalent circuit for a complex network.

Before we examine the steps involved in applying this theorem, it is important that an additional word be included here to ensure that the implications of the Thévenin equivalent circuit are clear. In Fig., the entire network, except  $R_L$ , is to be replaced by a single series resistor and battery as shown in Fig.. The values of these two elements of the Thévenin equivalent circuit must be chosen to ensure that the resistor  $R_L$  will react to the network of Fig., in the same manner as to the network of Fig.. In other words, the current through or voltage across  $R_L$  must be the same for either network for any value of  $R_L$ .

The following sequence of steps will lead to the proper value of  $R_{Th}$  and  $E_{Th}$ .

**Preliminary:**

1. Remove that portion of the network across which the Thévenin equivalent circuit is to be found. In Fig., this requires that the load resistor  $R_L$  be temporarily removed from the network.
2. Mark the terminals of the remaining two-terminal network. (The importance of this step will become obvious as we progress through some complex networks.)

$R_{Th}$ :

3. Calculate  $R_{Th}$  by first setting all sources to zero (voltage sources are replaced by short circuits, and current sources by open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.)

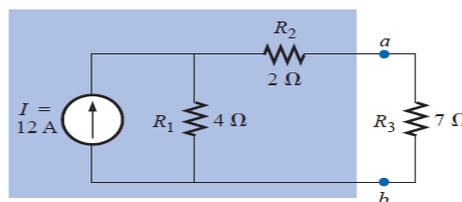
$E_{Th}$ :

4. Calculate  $E_{Th}$  by first returning all sources to their original position and finding the open-circuit voltage between the marked terminals. (This step is invariably the one that will lead to the most confusion and errors. In all cases, keep in mind that it is the open-circuit potential between the two terminals marked in step 2.)

Conclusion:

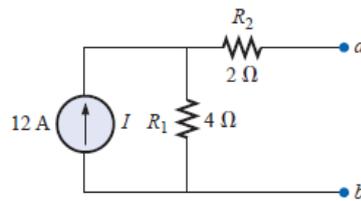
5. Draw the Thévenin equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit. This step is indicated by the placement of the resistor  $R_L$  between the terminals of the Thévenin equivalent circuit as shown in Fig..

**EXAMPLE :** Find the Thévenin equivalent circuit for the network in the shaded area of the network of Fig..



**Solution:**

Steps 1 and 2 are shown in Fig.



Establishing the terminals of particular interest for the network of Fig.

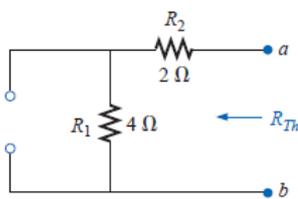
Step 3 is shown in Fig.. The current source has been replaced with an open-circuit equivalent, and the resistance determined between terminals a and b. In this case an ohmmeter connected between terminals a and b would send out a sensing current that would flow directly through R1 and R2 (at the same level). The result is that R1 and R2 are in series and the Thévenin resistance is the sum of the two.

$$R_{Th} = R_1 + R_2 = 4\Omega + 2\Omega = 6\Omega$$

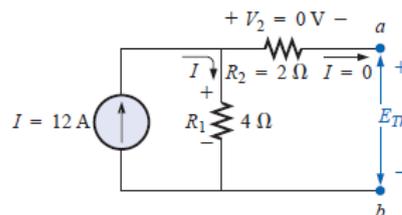
Step 4 : In this case, since an open circuit exists between the two marked terminals, the current is zero between these terminals and through the 2- resistor. The voltage drop across R2 is, therefore,

$$V_2 = I_2 R_2 = (0)R_2 = 0 \text{ V}$$

$$E_{Th} = V_1 = I_1 R_1 = IR_1 = (12 \text{ A})(4 \Omega) = 48 \text{ V}$$

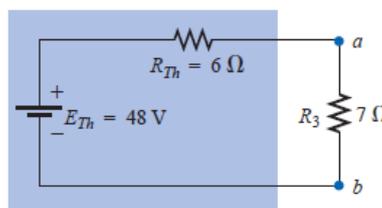


Determining  $R_{Th}$  for the network of Fig.



Determining  $E_{Th}$  for the network of Fig.

Step 5 is shown in Fig.



Substituting the Thévenin equivalent circuit in the network external to the resistor  $R_3$  of Fig.

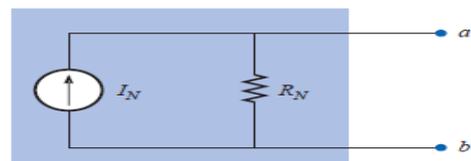
**NORTON'S THEOREM**

It was demonstrated that every voltage source with a series internal resistance has a current source equivalent. The current source equivalent of the Thévenin network, can be determined by **Norton's theorem**.

The theorem states the following:

Any two-terminal linear bilateral dc network can be replaced by an equivalent circuit consisting of a current source and a parallel resistor, as shown in Fig..

The discussion of Thévenin's theorem with respect to the equivalent circuit can also be applied to the Norton equivalent circuit. The steps leading to the proper values of  $I_N$  and  $R_N$  are now listed.



Norton equivalent circuit.

Preliminary:

1. Remove that portion of the network across which the Norton equivalent circuit is found.
2. Mark the terminals of the remaining two-terminal network.

$R_N$ :

3. Calculate  $R_N$  by first setting all sources to zero (voltage sources are replaced with short circuits, and current sources with open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.) Since  $R_N = R_{Th}$ , the procedure and value obtained using the approach described for Thévenin's theorem will determine the proper value of  $R_N$ .

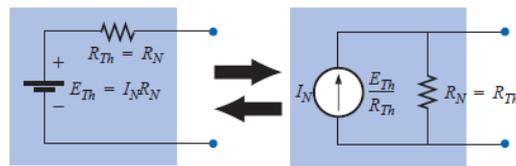
$I_N$ :

4. Calculate  $I_N$  by first returning all sources to their original position and then finding the short-circuit current between the marked terminals. It is the same current that would be measured by an ammeter placed between the marked terminals.

Conclusion:

5. Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit.

The Norton and Thévenin equivalent circuits can also be found from each other by using the source transformation discussed earlier in this chapter and reproduced in Fig.

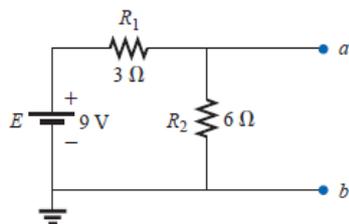
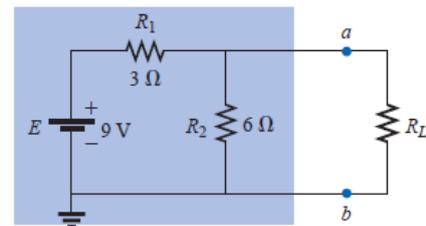


Converting between Thévenin and Norton equivalent circuits.

**EXAMPLE** Find the Norton equivalent circuit for the network in the shaded area of Fig.

**Solution:**

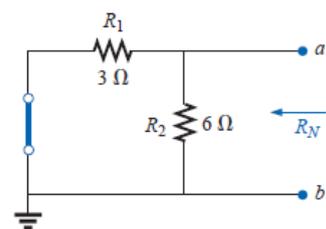
Steps 1 and 2 are shown in Fig.



Identifying the terminals of particular interest for the network of Fig.

Step 3 is shown in Fig. and

$$R_N = R_1 \parallel R_2 = 3 \Omega \parallel 6 \Omega = \frac{(3 \Omega)(6 \Omega)}{3 \Omega + 6 \Omega} = \frac{18 \Omega}{9} = 2 \Omega$$



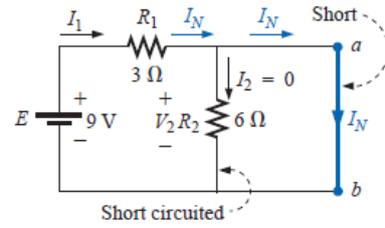
Determining  $R_N$  for the network of Fig.

Step 4 is shown in Fig. 9.63, clearly indicating that the short-circuit connection between terminals  $a$  and  $b$  is in parallel with  $R_2$  and eliminates its effect.  $I_N$  is therefore the same as through  $R_1$ , and the full battery voltage appears across  $R_1$  since

$$V_2 = I_2 R_2 = (0)6 \Omega = 0 \text{ V}$$

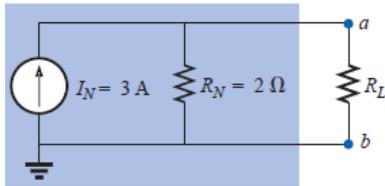
Therefore,

$$I_N = \frac{E}{R_1} = \frac{9 \text{ V}}{3 \Omega} = 3 \text{ A}$$

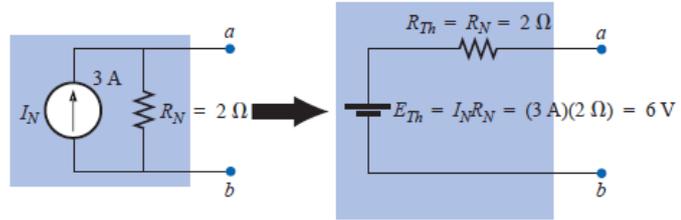


Determining  $I_N$  for the network of Fig.

Step 5: See Fig. This circuit is the same as the first one considered in the development of Thévenin's theorem. A simple conversion indicates that the Thévenin circuits are, in fact, the same



Substituting the Norton equivalent circuit for the network external to the resistor  $R_L$  of Fig.



Converting the Norton equivalent circuit of Fig. to a Thévenin equivalent circuit.

# Module III

## System Poles and Zeros:

The transfer function provides a basis for determining important system response characteristics without solving the complete differential equation. As defined, the transfer function is a rational function in the complex variable  $s = \sigma + j\omega$ , that is

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} \quad [1]$$

It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors

$$H(s) = \frac{N(s)}{D(s)} = K \frac{(s - z_1)(s - z_2)\dots(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_{n-1})(s - p_n)} \quad [2]$$

Where the numerator and denominator polynomials,  $N(s)$  and  $D(s)$ , have real coefficients defined by the system's differential equation and  $K = b_m/a_n$ . As written in Eq. (2) the  $z_i$ 's are the roots of the equation

$$N(s) = 0$$

and are defined to be the system zeros, and the  $p_i$ 's are the roots of the equation

$$D(s) = 0,$$

and are defined to be the system *poles*. In Eq. (2) the factors in the numerator and denominator are written so that when  $s = z_i$  the numerator  $N(s) = 0$  and the transfer function vanishes, that is

$$\lim_{s \rightarrow z_i} H(s) = 0$$

and similarly when  $s = p_i$  the denominator polynomial  $D(s) = 0$  and the value of the transfer function becomes unbounded,

$$\lim_{s \rightarrow p_i} H(s) = \infty$$

All of the coefficients of polynomials  $N(s)$  and  $D(s)$  are real, therefore the poles and zeros must be either purely real, or appear in complex conjugate pairs. In general for the poles, either  $p_i = \sigma_i$ , or else  $p_i, p_{i+1} = \sigma_i + j\omega_i$ . The existence of a single complex pole without a corresponding conjugate pole would generate complex coefficients in the polynomial  $D(s)$ . Similarly, the system zeros are either real or appear in complex conjugate pairs.

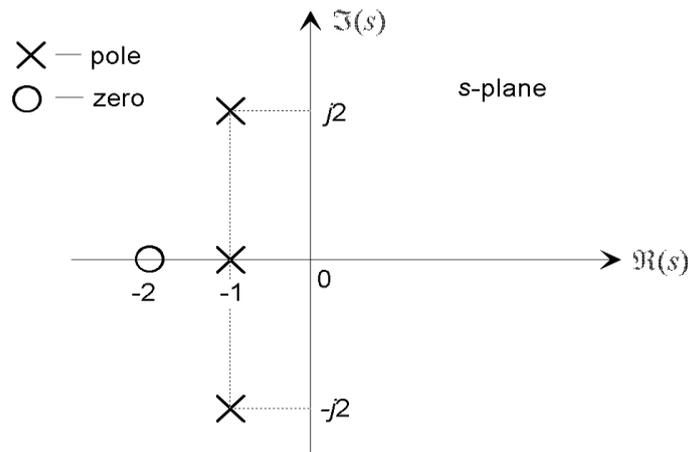


Figure 1: The pole-zero plot for a typical third-order system with one real pole and a complex conjugate pole pair, and a single real zero.

## The Pole-Zero Plot

A system is characterized by its poles and zeros in the sense that they allow reconstruction of the input/output differential equation. In general, the poles and zeros of a transfer function may be complex, and the system dynamics may be represented graphically by plotting their locations on the complex  $s$ -plane, whose axes represent the real and imaginary parts of the complex variable  $s$ . Such plots are known as *pole-zero plots*. It is usual to mark a zero location by a circle (°) and a pole location a cross (×). The location of the poles and zeros provide qualitative insights into the response characteristics of a system.

## **System Stability:**

The stability of a linear system may be determined directly from its transfer function. An  $n$ th order linear system is asymptotically stable only if all of the components in the homogeneous response from a finite set of initial conditions decay to zero as time increases, or

$$\lim_{t \rightarrow \infty} \sum_{i=1}^n c_i e^{p_i t} = 0$$

Where the  $p_i$  are the system poles. In a stable system all components of the homogeneous response must decay to zero as time increases. If any pole has a positive real part there is a component in the output that increases without bound, causing the system to be unstable.

In order for a linear system to be stable, all of its poles must have negative real parts that are they must all lie within the left-half of the  $s$ -plane. An “unstable” pole, lying in the right half of the  $s$ -plane, generates a component in the system homogeneous response that increases without bound from any finite initial conditions. A system having one or more poles lying on the imaginary axis of the  $s$ -plane has non-decaying oscillatory components in its homogeneous response, and is defined to be *marginally* stable.

## MODULE-IV

### TWO PORT NETWORK:

A pair of terminals through which a current may enter or leave a network is known as a *port*. Two-terminal devices or elements (such as resistors, capacitors, and inductors) result in one-port networks. Most of the circuits we have dealt with so far are two-terminal or one-port circuits, represented in Figure 2(a). We have considered the voltage across or current through a single pair of terminals—such as the two terminals of a resistor, a capacitor, or an inductor. We have also studied four-terminal or two-port circuits involving op amps, transistors, and transformers, as shown in Figure 2(b). In general, a network may have  $n$  ports. A port is an access to the network and consists of a pair of terminals; the current entering one terminal leaves through the other terminal so that the net current entering the port equals zero.

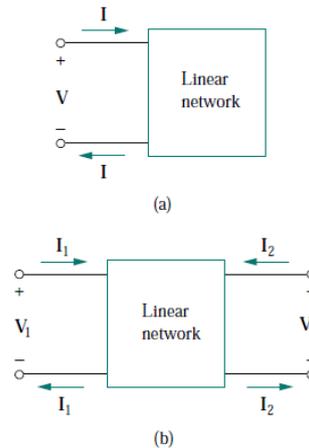


Figure 2: (a) One-port network, (b) two-port network.

Thus, a two-port network has two terminal pairs acting as access points. As shown in Figure 2(b), the current entering one terminal of a pair leaves the other terminal in the pair. Three-terminal devices such as transistors can be configured into two-port networks.

Our study of two-port networks is for at least two reasons. First, such networks are useful in communications, control systems, power systems, and electronics. For example, they are used in electronics to model transistors and to facilitate cascaded design. Second, knowing the parameters of a two-port network enables us to treat it as a “black box” when embedded within a larger network.

To characterize a two-port network requires that we relate the terminal quantities  $V_1$ ,  $V_2$ ,  $I_1$ , and  $I_2$  in Figure 2 (b), out of which two are independent. The various terms that relate these voltages and currents are called *parameters*. Our goal in this chapter is to derive six sets of these parameters. We will show the relationship between these parameters and how two-port networks can be connected in series, parallel, or cascade. As with op amps, we are only interested in the terminal behavior of the circuits. And we will assume that the two-port circuits contain no independent sources, although they can contain dependent sources. Finally, we will apply some of the concepts developed in this chapter to the analysis of transistor circuits and synthesis of ladder networks.

### IMPEDANCE PARAMETERS:

Impedance and admittance parameters are commonly used in the synthesis of filters. They are also useful in the design and analysis of impedance-matching networks and power distribution networks. We discuss impedance parameters in this section and admittance parameters in the next section.

A two-port network may be voltage-driven as in Figure 3 (a) or current-driven as in Figure 3(b). From either Figure 3(a) or (b), the terminal voltages can be related to the terminal currents as

$$\begin{aligned}V_1 &= Z_{11}I_1 + Z_{12}I_2 \\V_2 &= Z_{21}I_1 + Z_{22}I_2\end{aligned}$$

Or in matrix form as

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [z] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Where the  $z$  terms are called the *impedance parameters*, or simply  $z$  *parameters*, and have units of ohms.

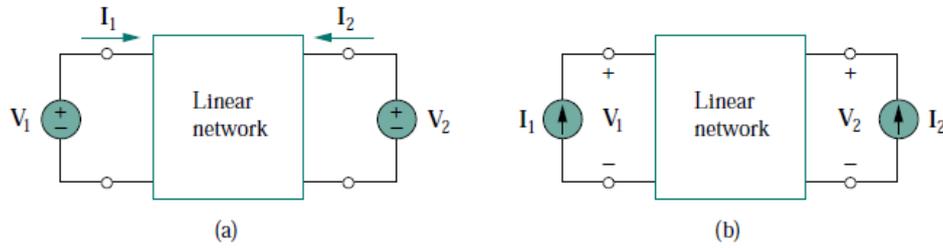


Figure 3: The linear two-port network: (a) driven by voltage sources, (b) driven by current

The values of the parameters can be evaluated by setting  $I_1 = 0$  (input port open-circuited) or  $I_2 = 0$  (output port open-circuited).

$$\begin{aligned} z_{11} &= \left. \frac{V_1}{I_1} \right|_{I_2=0}, & z_{12} &= \left. \frac{V_1}{I_2} \right|_{I_1=0} \\ z_{21} &= \left. \frac{V_2}{I_1} \right|_{I_2=0}, & z_{22} &= \left. \frac{V_2}{I_2} \right|_{I_1=0} \end{aligned}$$

Since the  $z$  parameters are obtained by open-circuiting the input or output port, they are also called the *open-circuit impedance parameters*. Specifically,

- $z_{11}$  = Open-circuit input impedance
- $z_{12}$  = Open-circuit transfer impedance from port 1 to port 2
- $z_{21}$  = Open-circuit transfer impedance from port 2 to port 1
- $z_{22}$  = Open-circuit output impedance

We obtain  $z_{11}$  and  $z_{21}$  by connecting a voltage  $V_1$  (or a current source  $I_1$ ) to port 1 with port 2 open-circuited as in Figure 4 and finding  $I_1$  and  $V_2$ ; we then get

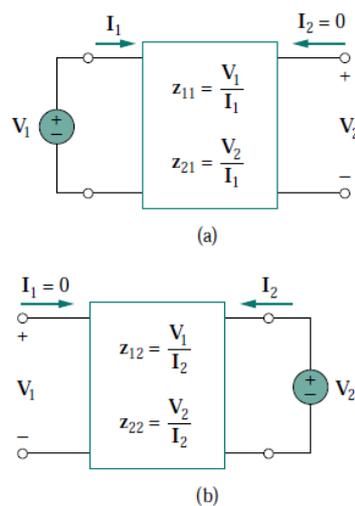


Figure 4: Determination of the  $z$  parameters: (a) finding  $z_{11}$  and  $z_{21}$  (b) finding  $z_{12}$  and  $z_{22}$ .

$$Z_{21} = \frac{V_2}{I_1}, \quad Z_{21} = \frac{V_2}{I_1}$$

we obtain  $\mathbf{z}_{12}$  and  $\mathbf{z}_{22}$  by connecting a voltage  $\mathbf{V}_2$  (or a current source  $\mathbf{I}_2$ ) to port 2 with port 1 open-circuited as in Figure 4) and finding  $\mathbf{I}_2$  and  $\mathbf{V}_1$ ; we then get

$$Z_{12} = \frac{V_1}{I_2}, \quad Z_{22} = \frac{V_2}{I_2}$$

The above procedure provides us with a means of calculating or measuring the  $z$  parameters.

Sometimes  $\mathbf{z}_{11}$  and  $\mathbf{z}_{22}$  are called *driving-point impedances*, while  $\mathbf{z}_{21}$  and  $\mathbf{z}_{12}$  are called *transfer impedances*. A driving-point impedance is the input impedance of a two-terminal (one-port) device. Thus,  $\mathbf{z}_{11}$  is the input driving-point impedance with the output port open-circuited, while  $\mathbf{z}_{22}$  is the output driving-point impedance with the input port open circuited.

When  $\mathbf{z}_{11} = \mathbf{z}_{22}$ , the two-port network is said to be *symmetrical*. This implies that the network has mirror like symmetry about some center line; that is, a line can be found that divides the network into two similar halves. When the two-port network is linear and has no dependent sources, the transfer impedances are equal ( $\mathbf{z}_{12} = \mathbf{z}_{21}$ ), and the two-port is said to be *reciprocal*. This means that if the points of excitation and response are interchanged, the transfer impedances remain the same. A two-port is reciprocal if interchanging an ideal voltage source at one port with an ideal ammeter at the other port gives the same ammeter reading.

## ADMITTANCE PARAMETERS:

In the previous section we saw that impedance parameters may not exist for a two-port network. So there is a need for an alternative means of describing such a network. This need is met by the second set of parameters, which we obtain by expressing the terminal currents in terms of the terminal voltages. In either Figure 5(a) or (b), the terminal currents can be expressed in terms of the terminal voltages as

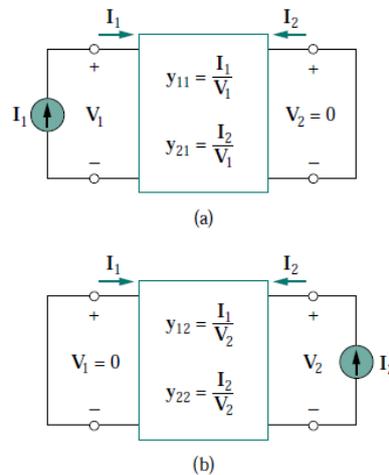


Figure 5: Determination of the  $y$  parameters: (a) finding  $y_{11}$  and  $y_{21}$ , (b) finding  $y_{12}$  and  $y_{22}$ .

$$\begin{aligned} I_1 &= Y_{11} V_1 + Y_{12} V_2 \\ I_2 &= Y_{21} V_1 + Y_{22} V_2 \end{aligned}$$

Or in matrix form as

$$\begin{aligned} \mathbf{I}_1 &= y_{11} \mathbf{V}_1 + y_{12} \mathbf{V}_2 \\ \mathbf{I}_2 &= y_{21} \mathbf{V}_1 + y_{22} \mathbf{V}_2 \end{aligned}$$

The  $\mathbf{y}$  terms are known as the *admittance parameters* (or, simply,  $y$  parameters) and have units of siemens.

The values of the parameters can be determined by setting  $V_1 = 0$  (input port short-circuited) or  $V_2 = 0$  (output port short-circuited). Thus,

$$\begin{aligned} y_{11} &= \frac{I_1}{V_1} \Big|_{V_2=0}, & y_{12} &= \frac{I_1}{V_2} \Big|_{V_1=0} \\ y_{21} &= \frac{I_2}{V_1} \Big|_{V_2=0}, & y_{22} &= \frac{I_2}{V_2} \Big|_{V_1=0} \end{aligned}$$

Since the  $y$  parameters are obtained by short-circuiting the input or output port, they are also called the *short-circuit admittance parameters*. Specifically,

- $y_{11}$  = Short-circuit input admittance
- $y_{12}$  = Short-circuit transfer admittance from port 2 to port 1
- $y_{21}$  = Short-circuit transfer admittance from port 1 to port 2
- $y_{22}$  = Short-circuit output admittance

We obtain  $y_{11}$  and  $y_{21}$  by connecting a current  $I_1$  to port 1 and short-circuiting port 2 as in Figure 5(a), finding  $V_1$

And  $I_2$ , and then calculating

$$y_{11} = \frac{I_1}{V_1}, \quad y_{21} = \frac{I_2}{V_1}$$

Similarly, we obtain  $y_{12}$  and  $y_{22}$  by connecting a current source  $I_2$  to port 2 and short-circuiting port 1 as in Figure 5(b), finding  $I_1$  and  $V_2$ , and then getting

$$y_{12} = \frac{I_1}{V_2}, \quad y_{22} = \frac{I_2}{V_2}$$

This procedure provides us with a means of calculating or measuring the  $y$  parameters. The impedance and admittance parameters are collectively referred to as *immittance* parameters.

### **HYBRID PARAMETERS:**

The  $z$  and  $y$  parameters of a two-port network do not always exist. So there is a need for developing another set of parameters. This third set of parameters is based on making  $V_1$  and  $I_2$  the dependent variables. Thus, we obtain

$$\begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ I_2 &= h_{21}I_1 + h_{22}V_2 \end{aligned}$$

Or in matrix form,

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = [h] \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

The  $h$  terms are known as the *hybrid parameters* (or, simply, *h parameters*) because they are a hybrid combination of ratios. They are very useful for describing electronic devices such as transistors; it is much easier to measure experimentally the  $h$  parameters of such devices than to measure their  $z$  or  $y$  parameters. The hybrid parameters are as follows.

$$\mathbf{h}_{11} = \left. \frac{\mathbf{V}_1}{\mathbf{I}_1} \right|_{\mathbf{V}_2=0}, \quad \mathbf{h}_{12} = \left. \frac{\mathbf{V}_1}{\mathbf{V}_2} \right|_{\mathbf{I}_1=0}$$

$$\mathbf{h}_{21} = \left. \frac{\mathbf{I}_2}{\mathbf{I}_1} \right|_{\mathbf{V}_2=0}, \quad \mathbf{h}_{22} = \left. \frac{\mathbf{I}_2}{\mathbf{V}_2} \right|_{\mathbf{I}_1=0}$$

It is evident that the parameters  $\mathbf{h}_{11}$ ,  $\mathbf{h}_{12}$ ,  $\mathbf{h}_{21}$ , and  $\mathbf{h}_{22}$  represent an impedance, a voltage gain, a current gain, and an admittance, respectively. This is why they are called the hybrid parameters. To be specific,

- $\mathbf{h}_{11}$  = Short-circuit input impedance
- $\mathbf{h}_{12}$  = Open-circuit reverse voltage gain
- $\mathbf{h}_{21}$  = Short-circuit forward current gain
- $\mathbf{h}_{22}$  = Open-circuit output admittance

The procedure for calculating the  $h$  parameters is similar to that used for the  $z$  or  $y$  parameters. We apply a voltage or current source to the appropriate port, short-circuit or open-circuit the other port, depending on the parameter of interest, and perform regular circuit analysis.

### TRANSMISSION PARAMETERS:

Since there are no restrictions on which terminal voltages and currents should be considered independent and which should be dependent variables, we expect to be able to generate many sets of parameters. Another set of parameters relates the variables at the input port to those at the output port. Thus,

$$\mathbf{V}_1 = \mathbf{A}\mathbf{V}_2 - \mathbf{B}\mathbf{I}_2$$

$$\mathbf{I}_1 = \mathbf{C}\mathbf{V}_2 - \mathbf{D}\mathbf{I}_2$$

Or

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix}$$

The above Equations are relating the input variables ( $\mathbf{V}_1$  and  $\mathbf{I}_1$ ) to the output variables ( $\mathbf{V}_2$  and  $-\mathbf{I}_2$ ). Notice that in computing the transmission parameters,  $-\mathbf{I}_2$  is used rather than  $\mathbf{I}_2$ , because the current is considered to be leaving the network, as shown in Figure 6. This is done merely for conventional reasons; when you cascade two-ports (output to input), it is most logical to think of  $\mathbf{I}_2$  as leaving the two-port. It is also customary in the power – industry to consider  $\mathbf{I}_2$  as leaving the two-port.

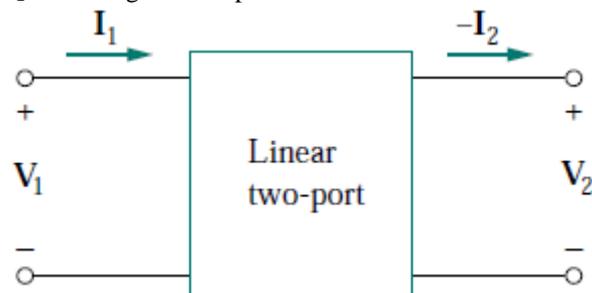


Figure 6: Terminal variables used to define the **ABCD** parameters.

The two-port parameters in above Eqs. provide a measure of how a circuit transmits voltage and current from a source to a load. They are useful in the analysis of transmission lines (such as cable and fiber) because they

express sending-end variables ( $V_1$  and  $I_1$ ) in terms of the receiving-end variables ( $V_2$  and  $-I_2$ ). For this reason, they are called *transmission parameters*. They are also known as **ABCD** parameters. They are used in the design of telephone systems, microwave networks, and radars.

The transmission parameters are determined as

$$\begin{aligned} \mathbf{A} &= \left. \frac{V_1}{V_2} \right|_{I_2=0}, & \mathbf{B} &= -\left. \frac{V_1}{I_2} \right|_{V_2=0} \\ \mathbf{C} &= \left. \frac{I_1}{V_2} \right|_{I_2=0}, & \mathbf{D} &= -\left. \frac{I_1}{I_2} \right|_{V_2=0} \end{aligned}$$

Thus, the transmission parameters are called, specifically,

**A** = Open-circuit voltage ratio

**B** = Negative short-circuit transfer impedance

**C** = Open-circuit transfer admittance

**D** = Negative short-circuit current ratio

**A** and **D** are dimensionless, **B** is in ohms, and **C** is in siemens. Since the transmission parameters provide a direct relationship between input and output variables, they are very useful in cascaded networks.

## RELATIONSHIPS BETWEEN PARAMETERS:

Since the six sets of parameters relate the same input and output terminal variables of the same two-port network, they should be interrelated. If two sets of parameters exist, we can relate one set to the other set. Let us demonstrate the process with two examples.

Given the  $z$  parameters, let us obtain the  $y$  parameters.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [z] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = [z]^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = [y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Comparing the above we have  $Y=[Z]^{-1}$ .

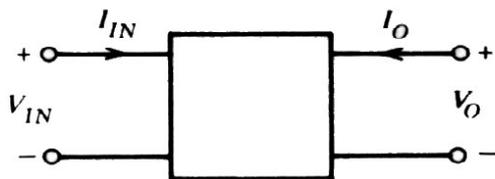
|          | <b>z</b>                   |                            | <b>y</b>                   |                            | <b>h</b>                   |                            | <b>g</b>                   |                            |
|----------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|
| <b>z</b> | $z_{11}$                   | $z_{12}$                   | $\frac{y_{22}}{\Delta_y}$  | $-\frac{y_{12}}{\Delta_y}$ | $\frac{\Delta_h}{h_{22}}$  | $\frac{h_{12}}{h_{22}}$    | $\frac{1}{g_{11}}$         | $-\frac{g_{12}}{g_{11}}$   |
|          | $z_{21}$                   | $z_{22}$                   | $-\frac{y_{21}}{\Delta_y}$ | $\frac{y_{11}}{\Delta_y}$  | $-\frac{h_{21}}{h_{22}}$   | $\frac{1}{h_{22}}$         | $\frac{g_{21}}{g_{11}}$    | $\frac{\Delta_g}{g_{11}}$  |
| <b>y</b> | $\frac{z_{22}}{\Delta_z}$  | $-\frac{z_{12}}{\Delta_z}$ | $y_{11}$                   | $y_{12}$                   | $\frac{1}{h_{11}}$         | $-\frac{h_{12}}{h_{11}}$   | $\frac{\Delta_g}{g_{22}}$  | $\frac{g_{12}}{g_{22}}$    |
|          | $-\frac{z_{21}}{\Delta_z}$ | $\frac{z_{11}}{\Delta_z}$  | $y_{21}$                   | $y_{22}$                   | $\frac{h_{21}}{h_{11}}$    | $\frac{\Delta_h}{h_{11}}$  | $-\frac{g_{21}}{g_{22}}$   | $\frac{1}{g_{22}}$         |
| <b>h</b> | $\frac{\Delta_z}{z_{22}}$  | $\frac{z_{12}}{z_{22}}$    | $\frac{1}{y_{11}}$         | $-\frac{y_{12}}{y_{11}}$   | $h_{11}$                   | $h_{12}$                   | $\frac{g_{22}}{\Delta_g}$  | $-\frac{g_{12}}{\Delta_g}$ |
|          | $-\frac{z_{21}}{z_{22}}$   | $\frac{1}{z_{22}}$         | $\frac{y_{21}}{y_{11}}$    | $\frac{\Delta_y}{y_{11}}$  | $h_{21}$                   | $h_{22}$                   | $-\frac{g_{21}}{\Delta_g}$ | $\frac{g_{11}}{\Delta_g}$  |
| <b>g</b> | $\frac{1}{z_{11}}$         | $-\frac{z_{12}}{z_{11}}$   | $\frac{\Delta_y}{y_{22}}$  | $\frac{y_{12}}{y_{22}}$    | $\frac{h_{22}}{\Delta_h}$  | $-\frac{h_{12}}{\Delta_h}$ | $g_{11}$                   | $g_{12}$                   |
|          | $\frac{z_{21}}{z_{11}}$    | $\frac{\Delta_z}{z_{11}}$  | $-\frac{y_{21}}{y_{22}}$   | $\frac{1}{y_{22}}$         | $-\frac{h_{21}}{\Delta_h}$ | $\frac{h_{11}}{\Delta_h}$  | $g_{21}$                   | $g_{22}$                   |

Table 1: Relation between parameters.

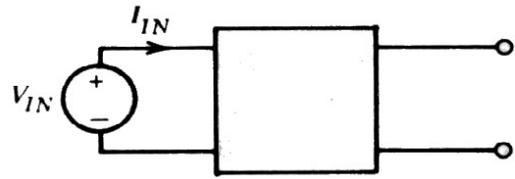
## MODULE-V

### Network Functions:

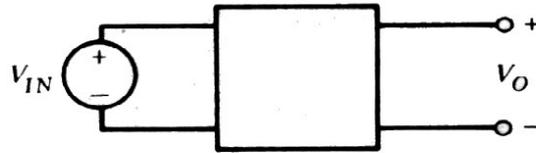
A **network function** is the Laplace transform of an impulse response. Its format is a ratio of two polynomials of the complex frequencies. Consider the general two-port network shown in Figure 2.2a. The terminal voltages and currents of the two-port can be related by two classes of network functions, namely, the driving point functions and the transfer functions.



(a) A two port network.



(b) Measuring input impedance.



(c) Measuring voltage gain.

The driving point functions relate the voltage at a port to the current at the same port. Thus, these functions are a property of a single port. For the input port the driving point impedance function  $Z_{IN}(s)$  is defined as:

This function can be measured by observing the current  $I_{IN}$  when the input port is driven by a voltage source  $V_{IN}$  (Figure 2.2b). The driving point admittance function  $Y_{IN}(s)$  is the reciprocal of the impedance function, and is given by:

$$Y_{IN}(s) = \frac{I_{IN}(s)}{V_{IN}(s)}$$

The output port driving point functions are defined in a similar way. The transfer functions of the two-port relate the voltage (or current) at one port to the voltage (or current) at the other port. The possible forms of transfer functions are:

1. The voltage transfer function, which is a ratio of one voltage to another voltage.

2. The current transfer function, which is a ratio of one current to another current.
3. The transfer impedance function, which is the ratio of a voltage to a current.
4. The transfer admittance function, which is the ratio of a current to a voltage.

The voltage transfer functions are defined with the output port an open circuit, as:

$$\text{voltage gain} = \frac{V_O(s)}{V_{IN}(s)}$$

$$\text{voltage loss (attenuation)} = \frac{V_{IN}(s)}{V_O(s)}$$

To evaluate the voltage gain, for example, the output voltage  $V_O$  is measured with the input port driven by a voltage source  $V_{IN}$  (Figure 2.2c). The other three types of transfer functions can be defined in a similar manner. Of the four types of transfer functions, the voltage transfer function is the one most often specified in the design of filters.

The functions defined above, when realized using resistors, inductors, capacitors, and active devices, can be shown to be the *ratios of polynomials in s* with real coefficients. This is so because the network functions are obtained by solving simple algebraic node equations, which involve at most the terms  $R$ ,  $sL$ ,  $sC$  and their reciprocals. The active device, if one exists, the solution still involves only the addition and multiplication of simple terms, which can only lead to a ratio of polynomials in  $s$ . In addition, all the coefficients of the numerator and denominator polynomials will be real. Thus, the general form of a network function is:

$$H(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_0}{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \cdots + b_0}$$

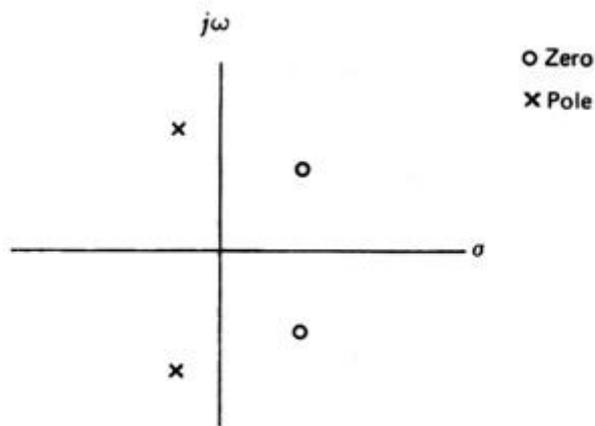
$$\text{where } a_n \neq 0 \quad b_m \neq 0$$

and all the coefficients  $a_i$  and  $b_i$  are real. If the numerator and denominator polynomials are factored, an alternate form of  $H(s)$  is obtained:

$$H(s) = \frac{a_n (s - z_1)(s - z_2) \cdots (s - z_n)}{b_m (s - p_1)(s - p_2) \cdots (s - p_m)}$$

In this expression  $z_1, z_2, \dots, z_n$  are called the zeros of  $H(s)$ , because  $H(s) = 0$  when  $s = z_i$ . The roots of the denominator  $p_1, p_2, \dots, p_m$  are called the poles of  $H(s)$ . It can be seen that  $H(s) = \infty$  at the poles,  $s = p_i$ .

The poles and zeros can be plotted on the complex  $s$  plane ( $s = \sigma + j\omega$ ), which has the real part  $\sigma$  for the abscissa, and the imaginary part  $j\omega$  for the ordinate (Figure 2.3).



Poles and zeros plotted in the complex  $s$  plane.

◆ **Properties of all Network Functions:**

We have already seen that **network functions are ratios of polynomials in  $s$  with real coefficients**. A consequence of this property is that **complex poles (and zeros) must occur in conjugate pairs**. To demonstrate this fact consider a complex root at ( $s = -a - jb$ ) which leads to the factor  $(s + a + jb)$  in the network function. The  $jb$  term will make some of the coefficients complex in the polynomial, unless the conjugate of the complex root at ( $s = -a + jb$ ) is also present in the polynomial. The product of a complex factor and its conjugate is

$$(s + a + jb)(s + a - jb) = s^2 + 2as + a^2 + b^2$$

which can be seen to have real coefficients.

Further important properties of network functions are obtained by restricting **the networks to be stable**, by which we mean that a bounded input excitation to the network must yield a bounded response. Put differently, the output of a stable network cannot be made to increase indefinitely by the application of a bounded input excitation. Passive networks are stable by their very nature, since they do not contain energy sources that might inject additional energy into the network. Active networks, however, do contain energy sources that could join forces with the input excitation to make the output increase

indefinitely. Such unstable networks, however, have no use in the world of practical filters and are therefore precluded from all our future discussions.

A convenient way of determining the stability of the general network function  $H(s)$

is by considering its response to an impulse function, which is obtained by taking the inverse Laplace transform of the partial fraction expansion of the function.

□□ If the network function has a simple pole on the real axis, the impulse response due to it (for  $t \geq 0$ ) will have the form:

$$h(t) = \mathcal{L}^{-1} \frac{K_1}{s - p_1} = K_1 e^{p_1 t}$$

For  $p_1$  positive, the impulse response is seen to increase exponentially with time, corresponding to an unstable circuit. Thus,  **$H(s)$  cannot have poles on the positive real axis.**

- Suppose  $H(s)$  has a pair of complex conjugate poles at  $s = a \pm jb$ . The contribution to the impulse response due to this pair of poles is

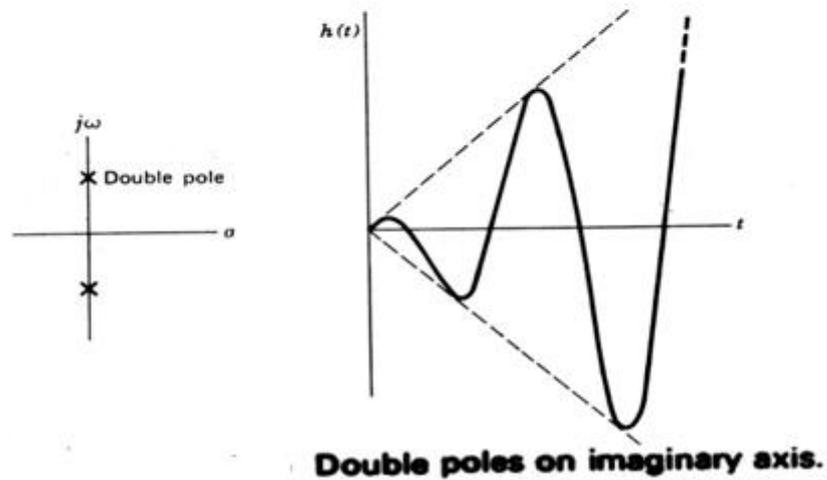
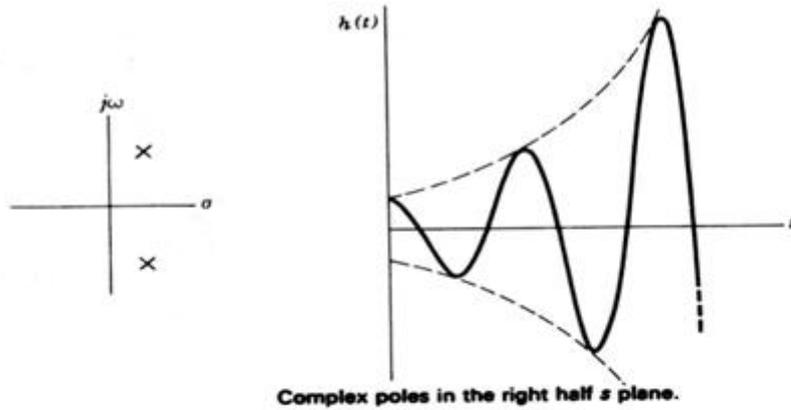
$$\begin{aligned} h(t) &= \mathcal{L}^{-1} \left( \frac{K_1}{s - a - jb} + \frac{K_1}{s - a + jb} \right) = \mathcal{L}^{-1} \frac{2K_1(s - a)}{(s - a)^2 + b^2} \\ &= 2K_1 e^{at} \cos bt \end{aligned}$$

Now if  $a$  is positive, corresponding to poles in the right half  $s$  plane, the response is seen to be an exponentially increasing sinusoid (Figure 2.4b). Therefore,  **$H(s)$  cannot have poles in the right half  $s$  plane.** An additional restriction on the poles of  $H(s)$  is that **any poles on the imaginary axis must be simple.**

Similarly, it can be shown that higher order poles on the  $j\omega$  axis will also cause the network to be unstable. From the above discussion we see that  $H(s)$  has the following factored form:

$$H(s) = \frac{N(s)}{\prod_i (s + a_i) \prod_k (s^2 + c_k s + d_k)}$$

where  $N(s)$  is the numerator polynomial and the constants associated with the denominator  $a_i$ ,  $c_k$ , and  $d_k$  are real and nonnegative. The  $(s + a_i)$  terms represent poles on the negative real axis and the second order terms represent complex conjugate poles in the left half  $s$  plane. It is easy to see that the product of these factors can only lead to a polynomial, all of whose coefficients are real and positive; moreover, none of the coefficients may be zero unless all the even or all the odd terms are missing.



*In summary*, the network functions of all passive networks and all stable active

Must be rational functions in s with real coefficients.

May not have poles in the right half s plane.

May not have multiple poles on the  $j\omega$  axis.

Example: Check to see whether the following are stable network functions:

$$(a) \frac{s}{s^2 - 3s + 4} \quad (b) \frac{s - 1}{s^2 + 4}$$

The first function cannot be realized by a stable network because one of the coefficients in the denominator polynomial is negative. It can easily be verified that the poles are in the right half s plane.

The second function is stable. The poles are on the  $j\omega$  axis (at  $s = \pm 2j$ ) and are simple. Note that the function has a zero in the right half  $s$  plane; however, this does not violate any of the requirements on network functions.

**Properties of Driving Point (Positive Real) Functions:**

*Positive Real Function:*

These conditions are required to satisfy to be positive realness

- $Y(s)$  must be a rational function in  $s$  with real coefficients, i.e., the coefficients of the numerator and denominator polynomials is real and positive.
  - The poles and zeros of  $Y(s)$  have either negative or zero real parts, i.e.,  $Y(s)$  not have poles or zeros in the right half  $s$  plane.
  - Poles of  $Y(s)$  on the imaginary axis must be simple and their residues must be real and positive, i.e.,  $Y(s)$  not has multiple poles or zeros on the  $j\omega$  axis. The same statement applies to the poles of  $1/Y(s)$ .
  - The degrees of the numerator and denominator polynomials in  $Y(s)$  differ at most by 1. Thus the number of finite poles and finite zeros of  $Y(s)$  differ at most by 1.
  - The terms of lowest degree in the numerator and denominator polynomials of  $Y(s)$  differ in degree at most by 1. So  $Y(s)$  has neither multiple poles nor zeros at the origin.
  - There be no missing terms in numerator and denominator polynomials unless all even or all odd terms are missing.
- ***Test for necessary and sufficient conditions:***
- $Y(s)$  must be real when  $s$  is real.
  - If  $Y(s) = p(s)/q(s)$ , then  $p(s) + q(s)$  must be Hurwitz. This requires that:
    - i. the continued fraction expansion of the Hurwitz test give only real and positive coefficients, and
    - ii. the continued fraction expansion not end prematurely.
  - In order that  $\text{Re} [Y(j\omega)] \geq 0$  for all  $\omega$ , it is necessary and sufficient that

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \Big|_{s=j\omega}$$

have no real positive roots of odd multiplicity. This may be determined by factoring  $A(\omega^2)$  or by the use of Sturm's theorem.



The real part test shows that:

$$m_1 m_2 - n_1 n_2 \Big|_{s=j\omega} = (2s^2 + 2)(s^2 + 1) \Big|_{s=j\omega} = 2(\omega^2 - 1)^2$$

which is always positive or zero as required. All of the tests are successful and the function  $F(s)$  is positive real.

### Brune's Positive Real Functions :

The term *positive-real function* was originally defined by Otto Brune to describe any function  $Z(s)$  which

- is rational (the quotient of two polynomials),
- is real when  $s$  is real
- has positive real part when  $s$  has a positive real part

### Properties of Positive Real

- The sum of two PR functions is PR.
- The composition of two PR functions is PR. In particular, if  $Z(s)$  is PR, then so are  $1/Z(s)$  and  $Z(1/s)$ .
- All the poles and zeros of a PR function are in the left half plane or on its boundary the imaginary axis.
- Any poles and zeroes on the imaginary axis are simple (have a multiplicity of one).
- Any poles on the imaginary axis have real strictly positive residues, and similarly at any zeroes on the imaginary axis, the function has a real strictly positive derivative.
- Over the right half plane, the minimum value of the real part of a PR function occurs on the imaginary axis (because the real part of an analytic function constitutes a harmonic function over the plane, and therefore satisfies the maximum principle).
- For a rational PR function, the number of poles and number of zeroes differ by at most one.

### Generalizations

#### Irrational functions

The irrational function  $Z(s)$  is PR if and only if

- $Z(s)$  is analytic in the open right half  $s$ -plane ( $\text{Re}[s] > 0$ )
- $Z(s)$  is real when  $s$  is positive and real
- $\text{Re}[Z(s)] \geq 0$  when  $\text{Re}[s] \geq 0$

#### Matrix-valued functions

A irrational matrix-valued function  $Z(s)$  is PR if and only if

- Each element of  $Z(s)$  is analytic in the open right half  $s$ -plane ( $\text{Re}[s] > 0$ )
- Each element of  $Z(s)$  is real when  $s$  is positive and real
- The Hermitian part  $(Z(s) + Z^\dagger(s))/2$  of  $Z(s)$  is positive semi-definite when  $\text{Re}[s] \geq 0$

## Maximum modulus principle :

The **maximum modulus principle** in complex analysis states that if  $f$  is a holomorphic function, then the modulus  $|f|$  cannot exhibit a true local maximum that is properly within the domain of  $f$ . In other words, either  $f$  is a constant function, or, for any point  $z_0$  inside the domain of  $f$  there exist other points arbitrarily close to  $z_0$  at which  $|f|$  takes larger values.

Let  $f$  be a function holomorphic on some connected open subset  $D$  of the complex plane  $\mathbb{C}$  and taking complex values. If  $z_0$  is a point in  $D$  such that

$$|f(z_0)| \geq |f(z)|$$

for all  $z$  in a neighborhood of  $z_0$ , then the function  $f$  is constant on  $D$ .

By switching to the reciprocal, we can get the **minimum modulus principle**. It states that if  $f$  is holomorphic within a bounded domain  $D$ , continuous up to the boundary of  $D$ , and non-zero at all points, then  $|f(z)|$  takes its minimum value on the boundary of  $D$ .

Alternatively, the maximum modulus principle can be viewed as a special case of the open mapping theorem, which states that a nonconstant holomorphic function maps open sets to open sets. If  $|f|$  attains a local maximum at  $z$ , then the image of a sufficiently small open neighborhood of  $z$  cannot be open. Therefore,  $f$  is constant.

### *Applications*

The maximum modulus principle has many uses in complex analysis, and may be used to prove the following:

- The fundamental theorem of algebra.
- Schwarz's lemma, a result which in turn has many generalisations and applications in complex analysis.
- The Phragmén–Lindelöf principle, an extension to unbounded domains.
- The Borel–Carathéodory theorem, which bounds an analytic function in terms of its real part.
- The Hadamard three-lines theorem, a result about the behaviour of bounded holomorphic functions on a line between two other parallel lines in the complex plane.

### *Hurwitz*

#### *Polynomial:*

A polynomial  $p(s)$  is said to be **Hurwitz** if all the roots of  $p(s)$  are located in the open left half (LH)  $s$ -plane (not including the imaginary axis).

Let  $p(s)$  be the polynomial in question. Assume first that  $p(s)$  is neither an even nor an odd polynomial. To test whether such a polynomial  $p(s)$  is indeed a Hurwitz polynomial, we may use *the Hurwitz test*.

- First decompose  $p(s)$  into its even and odd parts,  $M(s)$  and  $N(s)$ , respectively, as  $p(s) = M(s) + N(s)$ .
- Using  $M(s)$  and  $N(s)$  we form the test ratio  $T(s)$ , whose numerator has a higher degree than that of its denominator. Suppose that  $p(s)$  is a polynomial of degree  $d$ . Then

$$T(s) = \frac{N(s)}{M(s)} \quad \text{if } d \text{ is an odd integer} \quad (4-8a)$$

$$T(s) = \frac{M(s)}{N(s)} \quad \text{if } d \text{ is an even integer} \quad (4-8b)$$

- Next, we perform the continued fraction expansion about infinity on the test ratio  $T(s)$ , removing one pole at a time in the form of a quotient  $q_i$ , resulting in:

$$T(s) = q_1s + \frac{1}{q_2s + \frac{1}{q_3s + \frac{1}{\ddots + \frac{1}{q_d s}}}} \quad (4-9)$$

where  $q_i$  is the  $i$ th quotient, and  $q_i$  is the associated coefficient.

- If there is one or more quotients with negative coefficients, then  $p(s)$  is neither a Hurwitz nor a modified Hurwitz polynomial.
- On the other hand, if there are  $d$  quotients ( $d = \hat{d}$ ) and every quotient has a positive coefficient, then  $p(s)$  is a Hurwitz polynomial.
- Finally, if the number of quotient  $\hat{d}$  is less than  $d$  but every quotient has a positive coefficient, this means that there is a common factor  $k(s)$  between  $M(s)$  and  $N(s)$ . Hence, we can write  $p(s)$  as:

$$p(s) = k(s) [\hat{M}(s) + \hat{N}(s)] = k(s)\hat{p}(s) \quad (4-10)$$

where  $M(s) = k(s)\hat{M}(s)$ ,  $N(s) = k(s)\hat{N}(s)$ , and  $\hat{p}(s) = \hat{M}(s) + \hat{N}(s)$ .

- Because all the  $d^{\wedge}$  quotients of  $T(s)$  have positive coefficients, the polynomial  $p(s)$  in (4-10) is Hurwitz. Thus, if  $k(s)$  is a modified Hurwitz polynomial [i.e., if all the roots of  $k(s)$  are simple and purely imaginary], then  $p(s)$  is a modified Hurwitz polynomial.
- A procedure to determine if  $k(s)$  is a modified Hurwitz polynomial is described in the following in conjunction with the case when  $p(s)$  is either an even or an odd polynomial.
- Suppose now that  $p(s)$  is either an even or an odd polynomial of degree  $d$ .  $p(s)$  is a modified Hurwitz polynomial if and only if  $p(s)$  has only simple and imaginary axis roots (including the origin).
- To determine if  $p(s)$  is a modified Hurwitz polynomial, we form a test ratio  $T^{\wedge}(s)$ :

$$\hat{T}(s) = \frac{p(s)}{(d/ds)p(s)} = \frac{p(s)}{p'(s)} \quad (4-12)$$

and perform the continued fraction expansion about infinity on  $T^{\wedge}(s)$ , as in (4-9). Then  $p(s)$  is a modified Hurwitz polynomial if and only if there are  $d^{\wedge}$  quotients in the expansion and each quotient has a positive coefficient.

- In the case when  $p(s)$  is either an even or an odd polynomial, if there is one or more negative coefficient in the continued fraction expansion of  $T^{\wedge}(s)$ , then  $p(s)$  has a RH  $s$ -plane root; and if all coefficients are positive but there are only  $d^{\wedge} < d$  quotients, then all roots of  $p(s)$  are on the imaginary axis of the  $s$ -plane, but  $p(s)$  has non-simple or multiple roots. Either situation implies that  $p(s)$  is not a modified Hurwitz polynomial.

*The following examples illustrate these testing procedures*

**Example**      **Determine if**

$$p(s) = s^4 + 3s^3 + 5s^2 + 5s + 2 \quad (4-13)$$

**is a Hurwitz polynomial.**

**Solution:**  $p(s) = s^4 + 3s^3 + 5s^2 + 5s + 2$   
 $= (s^4 + 5s^2 + 2) + (3s^3 + 5s)$   
 $= M(s) + N(s)$

Because  $d = 4$  is even, the test ratio is

$$T(s) \triangleq \frac{M(s)}{N(s)} = \frac{s^4 + 5s^2 + 2}{3s^3 + 5s} \quad (4-14)$$

Clearly, at  $s = \infty$ ,  $T(s) = \infty$  [i.e.,  $T(s)$  has a pole at infinity]. Extracting this pole at infinity in the form of a quotient, we obtain

$$T(s) = \frac{1}{3}s + \frac{1}{T_1(s)} \quad (4-15)$$

where  $(1/3)s$  is the first quotient,  $1/3$  is its coefficient, and

$$\frac{1}{T_1(s)} = T(s) - \frac{1}{3}s = \frac{(10/3)s^2 + 2}{3s^3 + 5s}$$

is the remainder. Hence,

$$T_1(s) = \frac{3s^3 + 5s}{(10/3)s^2 + 2} \quad (4-16)$$

Observe that  $T_1(\infty) = \infty$ . Thus, we can extract a pole from  $T_1(s)$  in the form of a quotient as we did to  $T(s)$ . The result is to write  $T_1(s)$  as

$$T_1(s) = \frac{9}{10}s + \frac{1}{T_2(s)} \quad (4-17)$$

where  $(9/10)s$  is the second quotient,  $9/10$  is its coefficient, and  $1/T_2(s)$  is the second remainder. Substituting (4-17) into (4-11), we have

$$T(s) = \frac{1}{3}s + \frac{1}{(9/10)s + [1/T_2(s)]} \quad (4-18)$$

where

$$\frac{1}{T_2(s)} = T_1(s) - \frac{9}{10}s = \frac{(16/5)s}{(10/3)s^2 + 2}$$

or

$$T_2(s) = \frac{(10/3)s^2 + 2}{(16/5)s} \quad (4-19)$$

Clearly,  $T_2(\infty) = \infty$ . Removing the pole at infinity from  $T_2(s)$ , we obtain

$$T_2(s) = \frac{25}{24}s + \frac{1}{T_3(s)} \quad (4-20)$$

where  $(25/24)s$  is the third quotient,  $25/24$  is its coefficient, and

$$\frac{1}{T_3(s)} = T_2(s) - \frac{25}{24}s = \frac{2}{(16/5)s} = \frac{1}{(8/5)s} \quad (4-21)$$

is the third remainder. Substituting (4-20) and (4-21) into (4-18), we obtain the continued fraction expansion of  $T(s)$  at  $s = \infty$  as

$$T(s) = \frac{1}{3}s + \frac{1}{(9/10)s + \frac{1}{(25/24)s + \frac{1}{(8/5)s}}} \quad (4-22)$$

Because there are four quotients and their coefficients are positive (being  $1/3$ ,  $9/10$ ,  $25/24$ , and  $8/5$ ),  $p(s)$  is Hurwitz.

## Routh–Hurwitz stability criterion:

A tabular method can be used to determine the stability when the roots of a higher order characteristic polynomial are difficult to obtain. For an  $n$ th-degree polynomial

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

the table has  $n + 1$  rows and the following structure:

|           |           |           |          |
|-----------|-----------|-----------|----------|
| $a_n$     | $a_{n-2}$ | $a_{n-4}$ | ...      |
| $a_{n-1}$ | $a_{n-3}$ | $a_{n-5}$ | ...      |
| $b_1$     | $b_2$     | $b_3$     | ...      |
| $c_1$     | $c_2$     | $c_3$     | ...      |
| $\vdots$  | $\vdots$  | $\vdots$  | $\ddots$ |

where the elements  $b_i$  and  $c_i$  can be computed as follows:

$$b_i = \frac{a_{n-1} \times a_{n-2i} - a_n \times a_{n-2i-1}}{a_{n-1}}$$

$$c_i = \frac{b_1 \times a_{n-2i-1} - a_{n-1} \times b_{i+1}}{b_1}$$

When completed, the number of sign changes in the first column will be the number of non-negative poles.

In the first column, there are two sign changes, thus there are two non-negative roots where the system is unstable. Sometimes the presence of poles on the imaginary axis creates a situation of marginal stability. The row of polynomial which is just above the row containing the zeroes is called "Auxiliary Polynomial".

$$\bullet \quad s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0.$$

We have the following table:

|   |    |    |    |
|---|----|----|----|
| 1 | 8  | 20 | 16 |
| 2 | 12 | 16 | 0  |
| 2 | 12 | 16 | 0  |
| 0 | 0  | 0  | 0  |

In such a case the Auxiliary polynomial is  $A(s) = 2s^4 + 12s^2 + 16$ , which is again equal to zero. The next step is to differentiate the above equation which yields the following polynomial.  $B(s) = 8s^3 + 24s$ . The coefficients of the row containing zero now become "8" and "24". The process of Routh array is proceeded using these values which yield two points on the imaginary axis. These two points on the imaginary axis are the prime cause of marginal stability.

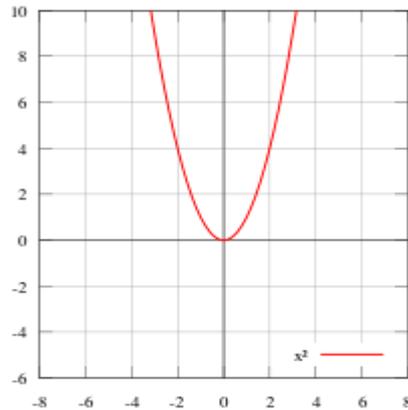
## **Even and Odd functions :**

In mathematics, **even functions** and **odd functions** are functions which satisfy particular symmetry relations, with respect to taking additive inverses. They are important in many areas of mathematical analysis, especially the theory of power series and Fourier series. They are named for the parity of the powers of the power functions which satisfy each condition: the function  $f(x) = x^n$  is an even function if  $n$  is an even integer, and it is an odd function if  $n$  is an odd integer.

### Definition and examples

The concept of evenness or oddness is only defined for functions whose domain and range both have an additive inverse. This includes additive groups, all rings, all fields, and all vector spaces. Thus, for example, a real-valued function of a real variable could be even or odd, as could a complex-valued function of a vector variable, and so on. The examples are real-valued functions of a real variable, to illustrate the symmetry of their graphs.

### Even functions



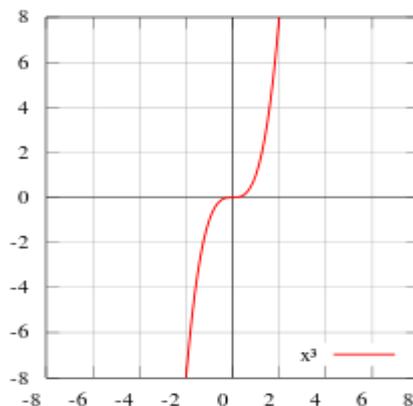
$f(x) = x^2$  is an example of an even function.

Let  $f(x)$  be a real-valued function of a real variable. Then  $f$  is **even** if the following equation holds for all  $x$  and  $-x$  in the domain of  $f$ :<sup>[1]</sup>

$$f(x) = f(-x), \quad f(x) - f(-x) = 0.$$

Geometrically speaking, the graph face of an even function is symmetric with respect to the  $y$ -axis, meaning that its graph remains unchanged after reflection about the  $y$ -axis. Examples of even functions are  $|x|$ ,  $x^2$ ,  $x^4$ ,  $\cos(x)$ , and  $\cosh(x)$ .

### Odd functions



$f(x) = x^3$  is an example of an odd function.

Again, let  $f(x)$  be a real-valued function of a real variable. Then  $f$  is **odd** if the following equation holds for all  $x$  and  $-x$  in the domain of  $f$ :<sup>[21]</sup>

$$-f(x) = f(-x), \quad f(x) + f(-x) = 0.$$

Geometrically, the graph of an odd function has rotational symmetry with respect to the origin, meaning that its graph remains unchanged after rotation of 180 degrees about the origin. Examples of odd functions are  $x$ ,  $x^3$ , sin( $x$ ), sinh( $x$ ), and erf( $x$ ).

### Continuity and differentiability

A function's being odd or even does not imply differentiability, or even continuity. For example, the Dirichlet function is even, but is nowhere continuous. Properties involving Fourier series, Taylor series, derivatives and so on may only be used when they can be assumed to exist.

## Sturm's theorem :

A **Sturm chain** or **Sturm sequence** is a finite sequence of polynomials  $p_0, p_1, \dots, p_m$

of decreasing degree with these following properties:

- $p_0 = p$  is square free (no square factors, i.e., no repeated roots);
- if  $p(\zeta) = 0$ , then  $\text{sign}(p_1(\zeta)) = \text{sign}(p'(\zeta))$ ;
- if  $p_i(\zeta) = 0$  for  $0 < i < m$  then  $\text{sign}(p_{i-1}(\zeta)) = -\text{sign}(p_{i+1}(\zeta))$ ;
- $p_m$  does not change its sign.

Sturm's sequence is a modification of Fourier's sequence. Let  $p = p_0, \dots, p_m$  be a Sturm chain, where  $p$  is a square-free polynomial, and let  $\sigma(\zeta)$  denote the number of sign changes (ignoring zeroes) in the sequence

$$p_0(\xi), p_1(\xi), p_2(\xi), \dots, p_m(\xi).$$

Sturm's theorem then states that for two real numbers  $a < b$ , the number of distinct roots of  $p$  in the half-open interval  $(a, b]$  is  $\sigma(a) - \sigma(b)$ . Suppose we wish to find the number of roots in some range for the polynomial  $p(x) = x^4 + x^3 - x - 1$ . So

$$\begin{aligned} p_0(x) &= p(x) = x^4 + x^3 - x - 1 \\ p_1(x) &= p'(x) = 4x^3 + 3x^2 - 1 \end{aligned}$$

Using polynomial long division to divide  $p_0$  by  $p_1$  gives the remainder  $-\frac{3}{16}x^2 - \frac{3}{4}x - \frac{15}{16}$ , and upon multiplying this remainder by  $-1$  we obtain  $p_2(x) = \frac{3}{16}x^2 + \frac{3}{4}x + \frac{15}{16}$ . Next dividing  $p_1$  by  $p_2$  and multiplying the remainder by  $-1$ , we obtain  $p_3(x) = -32x - 64$ . And dividing  $p_2$  by  $p_3$  and multiplying the remainder by  $-1$ , we obtain  $p_4(x) = -\frac{3}{16}$ .

Thus the complete chain of Sturm polynomials is:

$$p_0(x) = x^4 + x^3 - x - 1$$

$$p_1(x) = 4x^3 + 3x^2 - 1$$

$$p_2(x) = \frac{3}{16}x^2 + \frac{3}{4}x + \frac{15}{16}$$

$$p_3(x) = -32x - 64$$

$$p_4(x) = -\frac{3}{16}$$

To find the number of roots between  $-\infty$  and  $\infty$ , first evaluate  $p_0, p_1, p_2, p_3$ , and  $p_4$  at  $-\infty$  and note the sequence of signs of the results:  $+ - + + -$ , which contains three sign changes ( $+$  to  $-$ , then  $-$  to  $+$ , then  $+$  to  $-$ ). The same procedure for  $\infty$  gives the sign sequence  $+++--$ , which contains just one sign change. Hence the number of roots of the original polynomial between  $-\infty$  and  $\infty$  is  $3 - 1 = 2$ . That this is correct can be seen by noting that  $p(x) = x^4 + x^3 - x - 1$  can be factored as  $(x^2 - 1)(x^2 + x + 1)$ , where it is readily verifiable that  $x^2 - 1$  has the two roots  $-1$  and  $1$  while  $x^2 + x + 1$  has no real roots. In more complicated examples in which there is no advance knowledge of the roots because factoring is either impossible or impractical, one can experiment with various finite bounds for the range to be considered, thus narrowing down the locations of the roots.

## An alternative Test for Positive real functions:

Sturm's theorem can be used to compute the total number of real roots of a polynomial. This may be done by choosing  $-a = b = M$  where  $M$  is larger than the absolute value of every root. For example, a bound due to Cauchy says that all real roots of a polynomial with coefficients  $a_i$  are in the interval  $[-M, M]$ , where

$$M = 1 + \frac{\max_{i=0}^{n-1} |a_i|}{|a_n|}.$$

Although theoretically the above approach is the simplest, in practice bounds on the positive (only) roots of polynomials are used and the positive roots are isolated and evaluated first; the negative roots are treated by first substituting  $x$  by  $-x$ , then compute a new (positive root) bound to isolate and evaluate the negative roots. Sturm's method and VCA need to compute only one bound to isolate the positive roots. In contrast, to isolate the positive roots VAS needs to compute various positive bounds, for the various polynomials that appear in the process.<sup>[8][9]</sup>

Another method is computationally simpler. One can use the fact that for large  $x$ , the sign of

$$p(x) = a_n x^n + \dots$$

is  $\text{sign}(a_n)$ , whereas  $\text{sign}(p(-x))$  is  $(-1)^n a_n$ . In this way, simply counting the sign changes in the leading coefficients in the Sturm chain readily gives the number of distinct real roots of a polynomial.

Sturm's theorem allows also to determine the multiplicity of a given root, say  $\zeta$ . Indeed, suppose that  $a < \zeta < b$ , with  $\sigma(a) - \sigma(b) = 1$ . Then,  $\zeta$  has multiplicity  $k$  precisely when  $\zeta$  is a root with multiplicity  $k - 1$  of  $p_m$  (since it is the GCD of  $p$  and its derivative). Thus the multiplicity of  $\zeta$  may be computed by recursively applying Sturm's theorem to  $p_m$ . However, this method is rarely used, as square-free factorization is computationally more efficient for this purpose.

## MODULE-VI

### LC Network Synthesis :

By Kirchhoff's voltage law, the voltage across the capacitor,  $V_C$ , plus the voltage across the inductor,  $V_L$  must equal zero:

$$V_C + V_L = 0.$$

Likewise, by Kirchhoff's current law, the current through the capacitor equals the current through the inductor:

$$i_C = i_L.$$

From the constitutive relations for the circuit elements, we also know that

$$V_L(t) = L \frac{di_L}{dt}$$

and

$$i_C(t) = C \frac{dV_C}{dt}.$$

#### **Differential equation**

Rearranging and substituting gives the second order differential equation

$$\frac{d^2 i(t)}{dt^2} + \frac{1}{LC} i(t) = 0.$$

The parameter  $\omega_0$ , the resonant angular frequency, is defined as:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Using this can simplify the differential equation

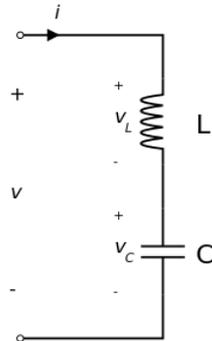
$$\frac{d^2 i(t)}{dt^2} + \omega_0^2 i(t) = 0.$$

The associated polynomial is

$$s^2 + \omega_0^2 = 0$$

Thus,  $s = +j\omega_0$      $s = -j\omega_0$  where  $j$  is the imaginary unit.

### **Series LC circuit**



Series LC circuit

In the series configuration of the LC circuit, the inductor  $L$  and capacitor  $C$  are connected in series, as shown here. The total voltage  $v$  across the open terminals is simply the sum of the voltage across the inductor and the voltage across the capacitor. The current  $i$  into the positive terminal of the circuit is equal to the current through both the capacitor and the inductor.

$$v = v_L + v_C$$

$$i = i_L = i_C$$

### **Resonance**

Inductive reactance magnitude ( $X_L$ ) increases as frequency increases while capacitive reactance magnitude ( $X_C$ ) decreases with the increase in frequency. At one particular frequency, these two reactances are equal in magnitude but opposite in sign; that frequency is called the resonant frequency ( $f_0$ ) for the given circuit.

Hence, at resonance:

$$X_L = -X_C$$

$$\omega L = \frac{1}{\omega C}$$

Solving for  $\omega$ , we have

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}}$$

which is defined as the resonant angular frequency of the circuit.

Converting angular frequency (in radians per second) into frequency (in hertz), one has

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

In a series configuration,  $X_C$  and  $X_L$  cancel each other out. In real, rather than idealised components, the current is opposed, mostly by the resistance of the coil windings. Thus, the current supplied to a series resonant circuit is a maximum at resonance.

- In the limit as  $f \rightarrow f_0$  current is maximum. Circuit impedance is minimum. In this state, a circuit is called an *acceptor circuit*
- For  $f < f_0$ ,  $X_L \ll (-X_C)$ . Hence, the circuit is capacitive.
- For  $f > f_0$ ,  $X_L \gg (-X_C)$ . Hence, the circuit is inductive.

### Impedance

In the series configuration, resonance occurs when the complex electrical impedance of the circuit approaches zero.

First consider the impedance of the series LC circuit. The total impedance is given by the sum of the inductive and capacitive impedances:

$$Z = Z_L + Z_C$$

Writing the inductive impedance as  $Z_L = j\omega L$  and capacitive impedance as  $Z_C = (j\omega C)^{-1}$  and substituting gives

$$Z(\omega) = j\omega L + \frac{1}{j\omega C}.$$

Writing this expression under a common denominator gives

$$Z(\omega) = j \frac{(\omega^2 LC - 1)}{\omega C}.$$

Finally, defining the natural angular frequency as

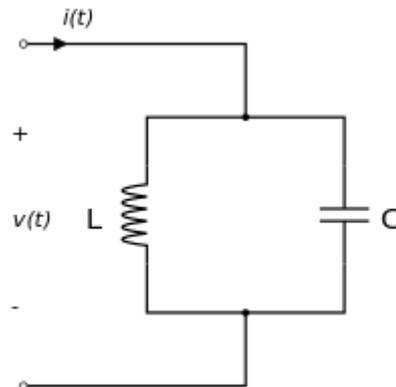
$$\omega_0 = \frac{1}{\sqrt{LC}},$$

the impedance becomes

$$Z(\omega) = jL \left( \frac{\omega^2 - \omega_0^2}{\omega} \right).$$

The numerator implies that in the limit as  $\omega \rightarrow \pm\omega_0$ , the total impedance  $Z$  will be zero and otherwise non-zero. Therefore the series LC circuit, when connected in series with a load, will act as a band-pass filter having zero impedance at the resonant frequency of the LC circuit.

### Parallel LC circuit



Parallel LC Circuit

In the parallel configuration, the inductor  $L$  and capacitor  $C$  are connected in parallel, as shown here. The voltage  $v$  across the open terminals is equal to both the voltage across the inductor and the voltage across the capacitor. The total current  $i$  flowing into the positive terminal of the circuit is equal to the sum of the current flowing through the inductor and the current flowing through the capacitor.

$$v = v_L = v_C$$

$$i = i_L + i_C$$

### Resonance

Let  $R$  be the internal resistance of the coil. When  $X_L$  equals  $X_C$ , the reactive branch currents are equal and opposite. Hence they cancel out each other to give minimum current in the main line. Since total current is minimum, in this state the total impedance is maximum.

Resonant frequency given by:

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

Note that any reactive branch current is not minimum at resonance, but each is given separately by dividing source voltage ( $V$ ) by reactance ( $Z$ ). Hence  $I=V/Z$ , as per Ohm's law.

- At  $f_0$ , line current is minimum. Total impedance is maximum. In this state a circuit is called a *rejector circuit*.
- Below  $f_0$ , circuit is inductive.
- Above  $f_0$ , circuit is capacitive.

### Impedance

The same analysis may be applied to the parallel LC circuit. The total impedance is then given by:

$$Z = \frac{Z_L Z_C}{Z_L + Z_C}$$

and after substitution of  $Z_L$  and  $Z_C$  and simplification, gives

$$Z(\omega) = -j \frac{\omega L}{\omega^2 LC - 1}$$

which further simplifies to

$$Z(\omega) = -j \left( \frac{1}{C} \right) \left( \frac{\omega}{\omega^2 - \omega_0^2} \right)$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Note that

$$\lim_{\omega \rightarrow \pm\omega_0} Z(\omega) = \infty$$

but for all other values of  $\omega$  the impedance is finite. The parallel LC circuit connected in series with a load will act as band-stop filter having infinite impedance at the resonant frequency of the LC circuit. The parallel LC circuit connected in parallel with a load will act as band-pass filter.

## **RC Network Synthesis:**

A **resistor–capacitor circuit (RC circuit)**, or **RC filter** or **RC network**, is an electric circuit composed of resistors and capacitors driven by a voltage or current source. A first order RC circuit is composed of one resistor and one capacitor and is the simplest type of RC circuit.

RC circuits can be used to filter a signal by blocking certain frequencies and passing others. The two most common RC filters are the high-pass filters and low-pass filters; band-pass filters and band-stop filters usually require RLC filters, though crude ones can be made with RC filters.

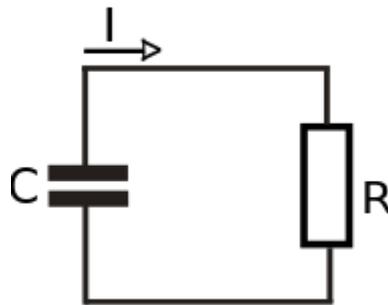
### ***Introduction***

There are three basic, linear passive lumped analog circuit components: the resistor (R), the capacitor (C), and the inductor (L). These may be combined in the RC circuit, the RL circuit, the LC circuit, and the RLC circuit, with the abbreviations indicating which components are

used. These circuits, among them, exhibit a large number of important types of behaviour that are fundamental to much of analog electronics. In particular, they are able to act as passive filters. This article considers the RC circuit, in both series and parallel forms, as shown in the diagrams below.

*This article relies on knowledge of the complex impedance representation of capacitors and on knowledge of the frequency domain representation of signals.*

### **Natural response**



RC circuit

The simplest RC circuit is a capacitor and a resistor in series. When a circuit consists of only a charged capacitor and a resistor, the capacitor will discharge its stored energy through the resistor. The voltage across the capacitor, which is time dependent, can be found by using Kirchhoff's current law, where the current charging the capacitor must equal the current through the resistor. This results in the linear differential equation

$$C \frac{dV}{dt} + \frac{V}{R} = 0 .$$

Solving this equation for  $V$  yields the formula for exponential decay:

$$V(t) = V_0 e^{-\frac{t}{RC}} ,$$

where  $V_0$  is the capacitor voltage at time  $t = 0$ .

$$\frac{V_0}{e}$$

The time required for the voltage to fall to  $\frac{V_0}{e}$  is called the RC time constant and is given by

$$\tau = RC .$$

### ***Complex impedance***

The complex impedance,  $Z_C$  (in ohms) of a capacitor with capacitance  $C$  (in farads) is

$$Z_C = \frac{1}{sC}$$

The complex frequency  $s$  is, in general, a complex number,

$$s = \sigma + j\omega$$

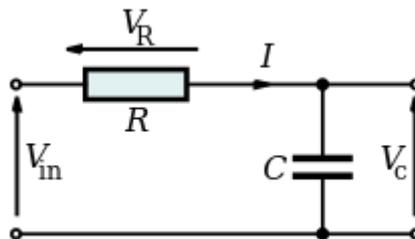
where

- $j$  represents the imaginary unit:

$$j^2 = -1$$

- $\sigma$  is the exponential decay constant (in radians per second), and
- $\omega$  is the sinusoidal angular frequency (also in radians per second).

### ***Series circuit***



Series RC circuit

By viewing the circuit as a voltage divider, the voltage across the capacitor is:

$$V_C(s) = \frac{1/Cs}{R + 1/Cs} V_{in}(s) = \frac{1}{1 + RCs} V_{in}(s)$$

and the voltage across the resistor is:

$$V_R(s) = \frac{R}{R + 1/Cs} V_{in}(s) = \frac{RCs}{1 + RCs} V_{in}(s)$$

### **Transfer functions**

The transfer function from the input voltage to the voltage across the capacitor is

$$H_C(s) = \frac{V_C(s)}{V_{in}(s)} = \frac{1}{1 + RCs}.$$

Similarly, the transfer function from the input to the voltage across the resistor is

$$H_R(s) = \frac{V_R(s)}{V_{in}(s)} = \frac{RCs}{1 + RCs}.$$

### Poles and zeros

Both transfer functions have a single pole located at

$$s = -\frac{1}{RC}.$$

In addition, the transfer function for the resistor has a zero located at the origin.

### Gain and phase

The magnitude of the gains across the two components are:

$$G_C = |H_C(j\omega)| = \left| \frac{V_C(j\omega)}{V_{in}(j\omega)} \right| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

and

$$G_R = |H_R(j\omega)| = \left| \frac{V_R(j\omega)}{V_{in}(j\omega)} \right| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}},$$

and the phase angles are:

$$\phi_C = \angle H_C(j\omega) = \tan^{-1}(-\omega RC)$$

and

$$\phi_R = \angle H_R(j\omega) = \tan^{-1}\left(\frac{1}{\omega RC}\right).$$

These expressions together may be substituted into the usual expression for the phasor representing the output:

$$V_C = G_C V_{in} e^{j\phi_C}$$

$$V_R = G_R V_{in} e^{j\phi_R}.$$

## Current

The current in the circuit is the same everywhere since the circuit is in series:

$$I(s) = \frac{V_{in}(s)}{R + \frac{1}{Cs}} = \frac{Cs}{1 + RCs} V_{in}(s)$$

## Impulse response

The impulse response for each voltage is the inverse Laplace transform of the corresponding transfer function. It represents the response of the circuit to an input voltage consisting of an impulse or Dirac delta function.

The impulse response for the capacitor voltage is

$$h_C(t) = \frac{1}{RC} e^{-t/RC} u(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

where  $u(t)$  is the Heaviside step function and

$$\tau = RC$$

is the time constant.

Similarly, the impulse response for the resistor voltage is

$$h_R(t) = \delta(t) - \frac{1}{RC} e^{-t/RC} u(t) = \delta(t) - \frac{1}{\tau} e^{-t/\tau} u(t)$$

where  $\delta(t)$  is the Dirac delta function

## Frequency-domain considerations

These are frequency domain expressions. Analysis of them will show which frequencies the circuits (or filters) pass and reject. This analysis rests on a consideration of what happens to these gains as the frequency becomes very large and very small.

$$\text{As } \omega \rightarrow \infty \quad G_C \rightarrow 0 \quad G_R \rightarrow 1.$$

As  $\omega \rightarrow 0$ :  $G_C \rightarrow 1$   $G_R \rightarrow 0$ . This shows that, if the output is taken across the capacitor, high frequencies are attenuated (shorted to ground) and low frequencies are passed. Thus, the circuit behaves as a low-pass filter. If, though, the output is taken across the resistor, high frequencies are passed and low frequencies are attenuated (since the capacitor blocks the signal as its frequency approaches 0). In this configuration, the circuit behaves as a high-pass filter.

The range of frequencies that the filter passes is called its bandwidth. The point at which the filter attenuates the signal to half its unfiltered power is termed its cutoff frequency. This requires that the gain of the circuit be reduced to

$$G_C = G_R = \frac{1}{\sqrt{2}}.$$

Solving the above equation yields

$$\omega_c = \frac{1}{RC}$$

or

$$f_c = \frac{1}{2\pi RC}$$

which is the frequency that the filter will attenuate to half its original power.

Clearly, the phases also depend on frequency, although this effect is less interesting generally than the gain variations.

As  $\omega \rightarrow 0$ :

$$\phi_C \rightarrow 0$$

$$\phi_R \rightarrow 90^\circ = \pi/2^c.$$

As  $\omega \rightarrow \infty$ :

$$\phi_C \rightarrow -90^\circ = -\pi/2^c$$

$$\phi_R \rightarrow 0$$

So at DC (0 Hz), the capacitor voltage is in phase with the signal voltage while the resistor voltage leads it by  $90^\circ$ . As frequency increases, the capacitor voltage comes to have a  $90^\circ$  lag relative to the signal and the resistor voltage comes to be in-phase with the signal.

### **Time-domain considerations**

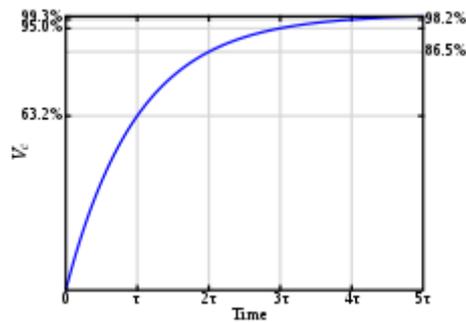
The most straightforward way to derive the time domain behaviour is to use the Laplace transforms of the expressions for  $V_C$  and  $V_R$  given above. This effectively transforms  $j\omega \rightarrow s$ . Assuming a step input (i.e.  $V_{in} = 0$  before  $t = 0$  and then  $V_{in} = V$  afterwards):

$$V_{in}(s) = V \frac{1}{s}$$

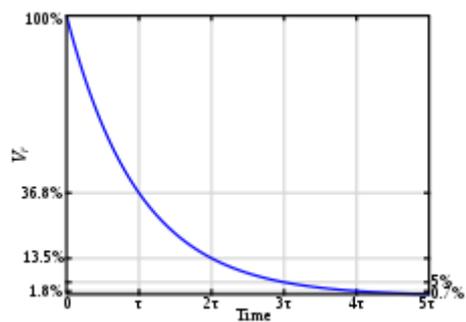
$$V_C(s) = V \frac{1}{1 + sRC} \frac{1}{s}$$

and

$$V_R(s) = V \frac{sRC}{1 + sRC} \frac{1}{s}$$



Capacitor voltage step-response.



Resistor voltage step-response.

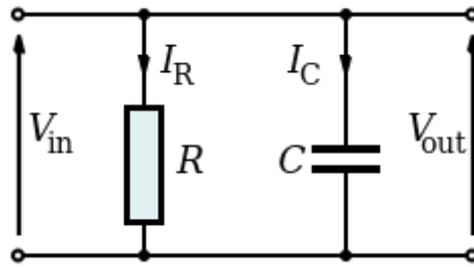
Partial fractions expansions and the inverse Laplace transform yield:

$$V_C(t) = V \left( 1 - e^{-t/RC} \right)$$

$$V_R(t) = V e^{-t/RC}$$

These equations are for calculating the voltage across the capacitor and resistor respectively while the capacitor is charging; for discharging, the equations are vice-versa. These equations can be rewritten in terms of charge and current using the relationships  $C=Q/V$  and  $V=IR$  (see Ohm's law).

### *Parallel circuit*



Parallel RC circuit

The parallel RC circuit is generally of less interest than the series circuit. This is largely because the output voltage  $V_{out}$  is equal to the input voltage  $V_{in}$ — as a result, this circuit does not act as a filter on the input signal unless fed by a current source.

With complex impedances:

$$I_R = \frac{V_{in}}{R}$$

and

$$I_C = j\omega CV_{in}.$$

This shows that the capacitor current is  $90^\circ$  out of phase with the resistor (and source) current. Alternatively, the governing differential equations may be used:

$$I_R = \frac{V_{in}}{R}$$

and

$$I_C = C \frac{dV_{in}}{dt}.$$

When fed by a current source, the transfer function of a parallel RC circuit is:

$$\frac{V_{out}}{I_{in}} = \frac{R}{1 + sRC}.$$

## **RL Network Synthesis:**

A **resistor-inductor circuit (RL circuit)**, or **RL filter** or **RL network**, is an electric circuit composed of resistors and inductors driven by a voltage or current source. A first order RL circuit is composed of one resistor and one inductor and is the simplest type of RL circuit.

The fundamental passive linear circuit elements are the resistor (R), capacitor (C) and inductor (L). These circuit elements can be combined to form an electrical circuit in four distinct ways: the RC circuit, the RL circuit, the LC circuit and the RLC circuit with the abbreviations indicating which components are used. These circuits exhibit important types of behaviour that are fundamental to analogue electronics. In particular, they are able to act as passive filters. This article considers the RL circuit in both series and parallel as shown in the diagrams.

In practice, however, capacitors (and RC circuits) are usually preferred to inductors since they can be more easily manufactured and are generally physically smaller, particularly for higher values of components.

Both RC and RL circuits form a single-pole filter. Depending on whether the reactive element (C or L) is in series with the load, or parallel with the load will dictate whether the filter is low-pass or high-pass.

Frequently RL circuits are used for DC power supplies to RF amplifiers, where the inductor is used to pass DC bias current and block the RF getting back into the power supply.

### ***Complex impedance***

The complex impedance  $Z_L$  (in ohms) of an inductor with inductance  $L$  (in henries) is

$$Z_L = Ls$$

The complex frequency  $s$  is a complex number,

$$s = \sigma + j\omega$$

where

- $j$  represents the imaginary unit:

$$j^2 = -1$$

- $\sigma$  is the exponential decay constant (in radians per second), and
- $\omega$  is the angular frequency (in radians per second).

### **Eigenfunctions**

The complex-valued eigenfunctions of *any* linear time-invariant (LTI) system are of the following forms:

$$\begin{aligned} \mathbf{V}(t) &= \mathbf{A}e^{st} = \mathbf{A}e^{(\sigma+j\omega)t} \\ \mathbf{A} &= Ae^{j\phi} \\ \Rightarrow \mathbf{V}(t) &= Ae^{j\phi}e^{(\sigma+j\omega)t} \\ &= Ae^{\sigma t}e^{j(\omega t+\phi)} \end{aligned}$$

From Euler's formula, the real-part of these eigenfunctions are exponentially-decaying sinusoids:

$$v(t) = \text{Re} \{V(t)\} = Ae^{\sigma t} \cos(\omega t + \phi)$$

### Sinusoidal steady state

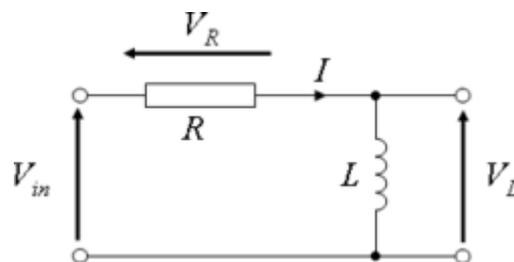
Sinusoidal steady state is a special case in which the input voltage consists of a pure sinusoid (with no exponential decay). As a result,

$$\sigma = 0$$

and the evaluation of  $s$  becomes

$$s = j\omega$$

### Series circuit



Series RL circuit

By viewing the circuit as a voltage divider, we see that the voltage across the inductor is:

$$V_L(s) = \frac{Ls}{R + Ls} V_{in}(s)$$

and the voltage across the resistor is:

$$V_R(s) = \frac{R}{R + Ls} V_{in}(s)$$

### Current

The current in the circuit is the same everywhere since the circuit is in series:

$$I(s) = \frac{V_{in}(s)}{R + Ls}$$

### Transfer functions

The transfer function for the inductor is

$$H_L(s) = \frac{V_L(s)}{V_{in}(s)} = \frac{Ls}{R + Ls} = G_L e^{j\phi_L}$$

Similarly, the transfer function for the resistor is

$$H_R(s) = \frac{V_R(s)}{V_{in}(s)} = \frac{R}{R + Ls} = G_R e^{j\phi_R}$$

### Poles and zeros

Both transfer functions have a single pole located at

$$s = -\frac{R}{L}$$

In addition, the transfer function for the inductor has a zero located at the origin.

### Gain and phase angle

The gains across the two components are found by taking the magnitudes of the above expressions:

$$G_L = |H_L(s)| = \left| \frac{V_L(s)}{V_{in}(s)} \right| = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}$$

and

$$G_R = |H_R(s)| = \left| \frac{V_R(s)}{V_{in}(s)} \right| = \frac{R}{\sqrt{R^2 + (\omega L)^2}}$$

and the phase angles are:

$$\phi_L = \angle H_L(s) = \tan^{-1} \left( \frac{R}{\omega L} \right)$$

and

$$\phi_R = \angle H_R(s) = \tan^{-1} \left( -\frac{\omega L}{R} \right)$$

### Phasor notation

These expressions together may be substituted into the usual expression for the phasor representing the output:

$$V_L = G_L V_{in} e^{j\phi_L}$$

$$V_R = G_R V_{in} e^{j\phi_R}$$

### Impulse response

The impulse response for each voltage is the inverse Laplace transform of the corresponding transfer function. It represents the response of the circuit to an input voltage consisting of an impulse or Dirac delta function.

The impulse response for the inductor voltage is

$$h_L(t) = \delta(t) - \frac{R}{L} e^{-t\frac{R}{L}} u(t) = \delta(t) - \frac{1}{\tau} e^{-\frac{1}{\tau}t} u(t)$$

where  $u(t)$  is the Heaviside step function and

$$\tau = \frac{L}{R}$$

is the time constant.

Similarly, the impulse response for the resistor voltage is

$$h_R(t) = \frac{R}{L} e^{-t\frac{R}{L}} u(t) = \frac{1}{\tau} e^{-\frac{1}{\tau}t} u(t)$$

### Zero input response (ZIR)

The **Zero input response**, also called the **natural response**, of an RL circuit describes the behavior of the circuit after it has reached constant voltages and currents and is disconnected from any power source. It is called the zero-input response because it requires no input.

The ZIR of an RL circuit is:

$$i(t) = i(0) e^{-\frac{R}{L}t} = i(0) e^{-\frac{1}{\tau}t}$$

### Frequency domain considerations

These are frequency domain expressions. Analysis of them will show which frequencies the circuits (or filters) pass and reject. This analysis rests on a consideration of what happens to these gains as the frequency becomes very large and very small.

As  $\omega \rightarrow \infty$ :

$$G_L \rightarrow 1$$

$$G_R \rightarrow 0$$

As  $\omega \rightarrow 0$ :

$$G_L \rightarrow 0$$

$$G_R \rightarrow 1$$

This shows that, if the output is taken across the inductor, high frequencies are passed and low frequencies are attenuated (rejected). Thus, the circuit behaves as a high-pass filter. If, though, the output is taken across the resistor, high frequencies are rejected and low frequencies are passed. In this configuration, the circuit behaves as a low-pass filter. Compare this with the behaviour of the resistor output in an RC circuit, where the reverse is the case.

The range of frequencies that the filter passes is called its bandwidth. The point at which the filter attenuates the signal to half its unfiltered power is termed its cutoff frequency. This requires that the gain of the circuit be reduced to

$$G_L = G_R = \frac{1}{\sqrt{2}}.$$

Solving the above equation yields

$$\omega_c = \frac{R}{L \text{ rad/s}}$$

or

$$f_c = \frac{R}{2\pi L \text{ Hz}}$$

which is the frequency that the filter will attenuate to half its original power.

Clearly, the phases also depend on frequency, although this effect is less interesting generally than the gain variations.

As  $\omega \rightarrow 0$ :

$$\phi_L \rightarrow 90^\circ = \frac{\pi}{2}$$

$$\phi_R \rightarrow 0$$

As  $\omega \rightarrow \infty$ :

$$\phi_L \rightarrow 0$$

$$\phi_R \rightarrow -90^\circ = -\frac{\pi}{2}$$

So at DC (0 Hz), the resistor voltage is in phase with the signal voltage while the inductor voltage leads it by  $90^\circ$ . As frequency increases, the resistor voltage comes to have a  $90^\circ$  lag relative to the signal and the inductor voltage comes to be in-phase with the signal.

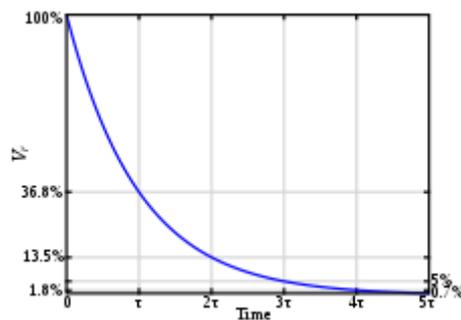
### Time domain considerations

The most straightforward way to derive the time domain behaviour is to use the Laplace transforms of the expressions for  $V_L$  and  $V_R$  given above. This effectively transforms  $j\omega \rightarrow s$ . Assuming a step input (i.e.,  $V_{in} = 0$  before  $t = 0$  and then  $V_{in} = V$  afterwards):

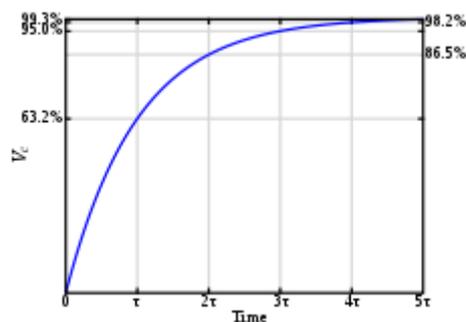
$$V_{in}(s) = V \frac{1}{s}$$

$$V_L(s) = V \frac{sL}{R + sL} \frac{1}{s}$$

$$V_R(s) = V \frac{R}{R + sL} \frac{1}{s}$$



Inductor voltage step-response.



## Resistor voltage step-response.

Partial fractions expansions and the inverse Laplace transform yield:

$$V_L(t) = V e^{-t\frac{R}{L}}$$
$$V_R(t) = V \left(1 - e^{-t\frac{R}{L}}\right)$$

Thus, the voltage across the inductor tends towards 0 as time passes, while the voltage across the resistor tends towards  $V$ , as shown in the figures. This is in keeping with the intuitive point that the inductor will only have a voltage across as long as the current in the circuit is changing — as the circuit reaches its steady-state, there is no further current change and ultimately no inductor voltage.

These equations show that a series RL circuit has a time constant, usually denoted  $\tau = \frac{L}{R}$  being the time it takes the voltage across the component to either fall (across L) or rise (across R) to within  $\frac{1}{e}$  of its final value. That is,  $\tau$  is the time it takes  $V_L$  to reach  $V\left(\frac{1}{e}\right)$  and  $V_R$  to reach  $V\left(1 - \frac{1}{e}\right)$ .

The rate of change is a *fractional*  $\left(1 - \frac{1}{e}\right)$  per  $\tau$ . Thus, in going from  $t = N\tau$  to  $t = (N + 1)\tau$ , the voltage will have moved about 63% of the way from its level at  $t = N\tau$  toward its final value. So the voltage across L will have dropped to about 37% after  $\tau$ , and essentially to zero (0.7%) after about  $5\tau$ . Kirchhoff's voltage law implies that the voltage across the resistor will *rise* at the same rate. When the voltage source is then replaced with a short-circuit, the voltage across R drops exponentially with  $t$  from  $V$  towards 0. R will be discharged to about 37% after  $\tau$ , and essentially fully discharged (0.7%) after about  $5\tau$ . Note that the current,  $I$ , in the circuit behaves as the voltage across R does, via Ohm's Law.

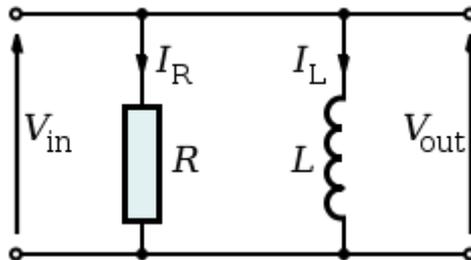
The delay in the rise/fall time of the circuit is in this case caused by the back-EMF from the inductor which, as the current flowing through it tries to change, prevents the current (and hence the voltage across the resistor) from rising or falling much faster than the time-constant of the circuit. Since all wires have some self-inductance and resistance, all circuits have a time constant. As a result, when the power supply is switched on, the current does not instantaneously reach its steady-state value,  $\frac{V}{R}$ . The rise instead takes several time-constants to complete. If this were not the case, and the current were to reach steady-state immediately, extremely strong inductive electric fields would be generated by the sharp change in the magnetic field — this would lead to breakdown of the air in the circuit and electric arcing, probably damaging components (and users).

These results may also be derived by solving the differential equation describing the circuit:

$$V_{in} = IR + L \frac{dI}{dt}$$
$$V_R = V_{in} - V_L$$

The first equation is solved by using an integrating factor and yields the current which must be differentiated to give  $V_L$ ; the second equation is straightforward. The solutions are exactly the same as those obtained via Laplace transforms.

**Parallel circuit**



Parallel RL circuit

The parallel RL circuit is generally of less interest than the series circuit unless fed by a current source. This is largely because the output voltage  $V_{out}$  is equal to the input voltage  $V_{in}$ —as a result, this circuit does not act as a filter for a voltage input signal.

With complex impedances:

$$I_R = \frac{V_{in}}{R}$$

$$I_L = \frac{V_{in}}{j\omega L} = -\frac{jV_{in}}{\omega L}$$

This shows that the inductor lags the resistor (and source) current by 90°.

The parallel circuit is seen on the output of many amplifier circuits, and is used to isolate the amplifier from capacitive loading effects at high frequencies. Because of the phase shift introduced by capacitance, some amplifiers become unstable at very high frequencies, and tend to oscillate. This affects sound quality and component life (especially the transistors), and is to be avoided.

**RLC Network Synthesis:**



A series RLC circuit: a resistor, inductor, and a capacitor

An **RLC circuit** (the letters R, L and C can be in other orders) is an electrical circuit consisting of a resistor, an inductor, and a capacitor, connected in series or in parallel. The RLC part of the name is due to those letters being the usual electrical symbols for resistance, inductance and capacitance respectively. The circuit forms a harmonic oscillator for current and will resonate in a similar way as an LC circuit will. The main difference that the presence of the resistor makes is that any oscillation induced in the circuit will die away over time if it is not kept going by a source. This effect of the resistor is called damping. The presence of the resistance also reduces the peak resonant frequency somewhat. Some resistance is unavoidable in real circuits, even if a resistor is not specifically included as a component. An ideal, pure LC circuit is an abstraction for the purpose of theory.

There are many applications for this circuit. They are used in many different types of oscillator circuits. Another important application is for tuning, such as in radio receivers or television sets, where they are used to select a narrow range of frequencies from the ambient radio waves. In this role the circuit is often referred to as a tuned circuit. An RLC circuit can be used as a band-pass filter, band-stop filter, low-pass filter or high-pass filter. The tuning application, for instance, is an example of band-pass filtering. The RLC filter is described as a *second-order* circuit, meaning that any voltage or current in the circuit can be described by a second-order differential equation in circuit analysis.

The three circuit elements can be combined in a number of different topologies. All three elements in series or all three elements in parallel are the simplest in concept and the most straightforward to analyse. There are, however, other arrangements, some with practical importance in real circuits. One issue often encountered is the need to take into account inductor resistance. Inductors are typically constructed from coils of wire, the resistance of which is not usually desirable, but it often has a significant effect on the circuit.

### ***Resonance***

An important property of this circuit is its ability to resonate at a specific frequency, the resonance frequency,  $f_0$ . Frequencies are measured in units of hertz. In this article, however, angular frequency,  $\omega_0$ , is used which is more mathematically convenient. This is measured in radians per second. They are related to each other by a simple proportion,

$$\omega_0 = 2\pi f_0$$

Resonance occurs because energy is stored in two different ways: in an electric field as the capacitor is charged and in a magnetic field as current flows through the inductor. Energy can be transferred from one to the other within the circuit and this can be oscillatory. A mechanical analogy is a weight suspended on a spring which will oscillate up and down when released. This is no passing metaphor; a weight on a spring is described by exactly the same second order differential equation as an RLC circuit and for all the properties of the one system there will be found an analogous property of the other. The mechanical property answering to the resistor in the circuit is friction in the spring/weight system. Friction will slowly bring any oscillation to a halt if there is no external force driving it. Likewise, the resistance in an RLC circuit will "damp" the oscillation, diminishing it with time if there is no driving AC power source in the circuit.

The resonance frequency is defined as the frequency at which the impedance of the circuit is at a minimum. Equivalently, it can be defined as the frequency at which the impedance is

purely real (that is, purely resistive). This occurs because the impedances of the inductor and capacitor at resonance are equal but of opposite sign and cancel out. Circuits where L and C are in parallel rather than series actually have a maximum impedance rather than a minimum impedance. For this reason they are often described as antiresonators, it is still usual, however, to name the frequency at which this occurs as the resonance frequency.

## Natural frequency

The resonance frequency is defined in terms of the impedance presented to a driving source. It is still possible for the circuit to carry on oscillating (for a time) after the driving source has been removed or it is subjected to a step in voltage (including a step down to zero). This is similar to the way that a tuning fork will carry on ringing after it has been struck, and the effect is often called ringing. This effect is the peak natural resonance frequency of the circuit and in general is not exactly the same as the driven resonance frequency, although the two will usually be quite close to each other. Various terms are used by different authors to distinguish the two, but resonance frequency unqualified usually means the driven resonance frequency. The driven frequency may be called the undamped resonance frequency or undamped natural frequency and the peak frequency may be called the damped resonance frequency or the damped natural frequency. The reason for this terminology is that the driven resonance frequency in a series or parallel resonant circuit has the value<sup>[1]</sup>

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

This is exactly the same as the resonance frequency of an LC circuit, that is, one with no resistor present. The resonant frequency for an RLC circuit is the same as a circuit in which there is no damping, hence undamped resonance frequency. The peak resonance frequency, on the other hand, depends on the value of the resistor and is described as the damped resonant frequency. A highly damped circuit will fail to resonate at all when not driven. A circuit with a value of resistor that causes it to be just on the edge of ringing is called critically damped. Either side of critically damped are described as underdamped (ringing happens) and overdamped (ringing is suppressed).

Circuits with topologies more complex than straightforward series or parallel (some examples described later in the article) have a driven resonance frequency that deviates from  $\omega_0 = \frac{1}{\sqrt{LC}}$  and for those the undamped resonance frequency, damped resonance frequency and driven resonance frequency can all be different.

## Damping

Damping is caused by the resistance in the circuit. It determines whether or not the circuit will resonate naturally (that is, without a driving source). Circuits which will resonate in this way are described as underdamped and those that will not are overdamped. Damping attenuation (symbol  $\alpha$ ) is measured in nepers per second. However, the unitless damping factor (symbol  $\zeta$ , zeta) is often a more useful measure, which is related to  $\alpha$  by

$$\zeta = \frac{\alpha}{\omega_0}$$

The special case of  $\zeta = 1$  is called critical damping and represents the case of a circuit that is just on the border of oscillation. It is the minimum damping that can be applied without causing oscillation.

## Bandwidth

The resonance effect can be used for filtering, the rapid change in impedance near resonance can be used to pass or block signals close to the resonance frequency. Both band-pass and band-stop filters can be constructed and some filter circuits are shown later in the article. A key parameter in filter design is bandwidth. The bandwidth is measured between the 3dB-points, that is, the frequencies at which the power passed through the circuit has fallen to half the value passed at resonance. There are two of these half-power frequencies, one above, and one below the resonance frequency

$$\Delta\omega = \omega_2 - \omega_1$$

where  $\Delta\omega$  is the bandwidth,  $\omega_1$  is the lower half-power frequency and  $\omega_2$  is the upper half-power frequency. The bandwidth is related to attenuation by,

$$\Delta\omega = 2\alpha$$

when the units are radians per second and nepers per second respectively<sup>[citation needed]</sup>. Other units may require a conversion factor. A more general measure of bandwidth is the fractional bandwidth, which expresses the bandwidth as a fraction of the resonance frequency and is given by

$$F_b = \frac{\Delta\omega}{\omega_0}$$

The fractional bandwidth is also often stated as a percentage. The damping of filter circuits is adjusted to result in the required bandwidth. A narrow band filter, such as a notch filter, requires low damping. A wide band filter requires high damping.

## Q factor

The Q factor is a widespread measure used to characterise resonators. It is defined as the peak energy stored in the circuit divided by the average energy dissipated in it per radian at resonance. Low  $Q$  circuits are therefore damped and lossy and high  $Q$  circuits are underdamped.  $Q$  is related to bandwidth; low  $Q$  circuits are wide band and high  $Q$  circuits are narrow band. In fact, it happens that  $Q$  is the inverse of fractional bandwidth

$$Q = \frac{1}{F_b} = \frac{\omega_0}{\Delta\omega}$$

Q factor is directly proportional to selectivity, as Q factor depends inversely on bandwidth.

For a series resonant circuit, the Q factor can be calculated as follows:<sup>[2]</sup>

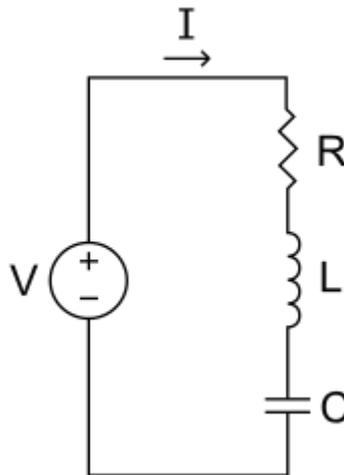
$$Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R}$$

### Scaled parameters

The parameters  $\zeta$ ,  $F_b$ , and  $Q$  are all scaled to  $\omega_0$ . This means that circuits which have similar parameters share similar characteristics regardless of whether or not they are operating in the same frequency band.

The article next gives the analysis for the series RLC circuit in detail. Other configurations are not described in such detail, but the key differences from the series case are given. The general form of the differential equations given in the series circuit section are applicable to all second order circuits and can be used to describe the voltage or current in any element of each circuit.

### Series RLC circuit



**Figure :** RLC series circuit

**V** – the voltage of the power source

**I** – the current in the circuit

**R** – the resistance of the resistor

**L** – the inductance of the inductor

**C** – the capacitance of the capacitor

In this circuit, the three components are all in series with the voltage source. The governing differential equation can be found by substituting into Kirchhoff's voltage law (KVL) the constitutive equation for each of the three elements. From KVL,

$$v_R + v_L + v_C = v(t)$$

where  $v_R, v_L, v_C$  are the voltages across R, L and C respectively and  $v(t)$  is the time varying voltage from the source. Substituting in the constitutive equations,

$$Ri(t) + L \frac{di}{dt} + \frac{1}{C} \int_{-\infty}^{\tau=t} i(\tau) d\tau = v(t)$$

For the case where the source is an unchanging voltage, differentiating and dividing by L leads to the second order differential equation:

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0$$

This can usefully be expressed in a more generally applicable form:

$$\frac{d^2i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0$$

$\alpha$  and  $\omega_0$  are both in units of angular frequency.  $\alpha$  is called the *neper frequency*, or *attenuation*, and is a measure of how fast the transient response of the circuit will die away after the stimulus has been removed. Neper occurs in the name because the units can also be considered to be nepers per second, neper being a unit of attenuation.  $\omega_0$  is the angular resonance frequency.<sup>[3]</sup>

For the case of the series RLC circuit these two parameters are given by:<sup>[4]</sup>

$$\alpha = \frac{R}{2L} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

A useful parameter is the *damping factor*,  $\zeta$  which is defined as the ratio of these two,

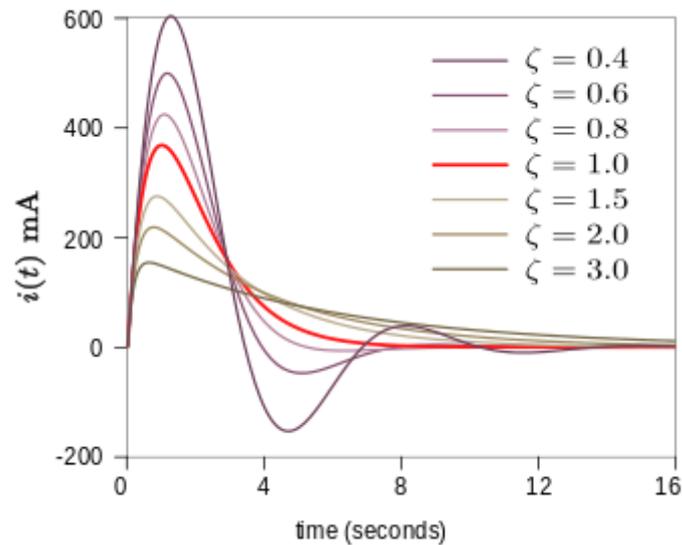
$$\zeta = \frac{\alpha}{\omega_0}$$

In the case of the series RLC circuit, the damping factor is given by,

$$\zeta = \frac{R}{2} \sqrt{\frac{C}{L}}$$

The value of the damping factor determines the type of transient that the circuit will exhibit.<sup>[5]</sup> Some authors do not use  $\zeta$  and call  $\alpha$  the damping factor.<sup>[6]</sup>

### Transient response



Plot showing underdamped and overdamped responses of a series RLC circuit. The critical damping plot is the bold red curve. The plots are normalised for  $L = 1$ ,  $C = 1$  and  $\omega_0=1$

The differential equation for the circuit solves in three different ways depending on the value of  $\zeta$ . These are underdamped ( $\zeta < 1$ ), overdamped ( $\zeta > 1$ ) and critically damped ( $\zeta = 1$ ). The differential equation has the characteristic equation,<sup>[7]</sup>

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

The roots of the equation in  $s$  are,<sup>[7]</sup>

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

The general solution of the differential equation is an exponential in either root or a linear superposition of both,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The coefficients  $A_1$  and  $A_2$  are determined by the boundary conditions of the specific problem being analysed. That is, they are set by the values of the currents and voltages in the circuit at the onset of the transient and the presumed value they will settle to after infinite time.<sup>[8]</sup>

### Overdamped response

The overdamped response ( $\zeta > 1$ ) is,<sup>[9]</sup>

$$i(t) = A_1 e^{-\omega_0(\zeta + \sqrt{\zeta^2 - 1})t} + A_2 e^{-\omega_0(\zeta - \sqrt{\zeta^2 - 1})t}$$

The overdamped response is a decay of the transient current without oscillation.<sup>[10]</sup>

### Underdamped response

The underdamped response ( $\zeta < 1$ ) is,<sup>[11]</sup>

$$i(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t)$$

By applying standard trigonometric identities the two trigonometric functions may be expressed as a single sinusoid with phase shift,<sup>[12]</sup>

$$i(t) = B_3 e^{-\alpha t} \sin(\omega_d t + \varphi)$$

The underdamped response is a decaying oscillation at frequency  $\omega_d$ . The oscillation decays at a rate determined by the attenuation  $\alpha$ . The exponential in  $\alpha$  describes the envelope of the oscillation.  $B_1$  and  $B_2$  (or  $B_3$  and the phase shift  $\varphi$  in the second form) are arbitrary constants determined by boundary conditions. The frequency  $\omega_d$  is given by,<sup>[11]</sup>

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0 \sqrt{1 - \zeta^2}$$

This is called the damped resonance frequency or the damped natural frequency. It is the frequency the circuit will naturally oscillate at if not driven by an external source. The resonance frequency,  $\omega_0$ , which is the frequency at which the circuit will resonate when driven by an external oscillation, may often be referred to as the undamped resonance frequency to distinguish it.<sup>[13]</sup>

### Critically damped response

The critically damped response ( $\zeta = 1$ ) is,<sup>[14]</sup>

$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

The critically damped response represents the circuit response that decays in the fastest possible time without going into oscillation. This consideration is important in control systems where it is required to reach the desired state as quickly as possible without overshooting.  $D_1$  and  $D_2$  are arbitrary constants determined by boundary conditions.<sup>[15]</sup>

### Laplace domain

The series RLC can be analyzed for both transient and steady AC state behavior using the Laplace transform.<sup>[16]</sup> If the voltage source above produces a waveform with Laplace-transformed  $V(s)$  (where  $s$  is the complex frequency  $s = \sigma + i\omega$ ), KVL can be applied in the Laplace domain:

$$V(s) = I(s) \left( R + Ls + \frac{1}{Cs} \right)$$

where  $I(s)$  is the Laplace-transformed current through all components. Solving for  $I(s)$ :

$$I(s) = \frac{1}{R + Ls + \frac{1}{Cs}} V(s)$$

And rearranging, we have that

$$I(s) = \frac{s}{L \left( s^2 + \frac{R}{L}s + \frac{1}{LC} \right)} V(s)$$

### Laplace admittance

Solving for the Laplace admittance  $Y(s)$ :

$$Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L \left( s^2 + \frac{R}{L}s + \frac{1}{LC} \right)}$$

Simplifying using parameters  $\alpha$  and  $\omega_0$  defined in the previous section, we have

$$Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L(s^2 + 2\alpha s + \omega_0^2)}$$

### Poles and zeros

The zeros of  $Y(s)$  are those values of  $s$  such that  $Y(s) = 0$ :

$$s = 0 \text{ and } |s| \rightarrow \infty$$

The poles of  $Y(s)$  are those values of  $s$  such that  $Y(s) \rightarrow \infty$ . By the quadratic formula, we find

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

The poles of  $Y(s)$  are identical to the roots  $s_1$  and  $s_2$  of the characteristic polynomial of the differential equation in the section above.

### General solution

For an arbitrary  $E(t)$ , the solution obtained by inverse transform of  $I(s)$  is:

$$I(t) = \frac{1}{L} \int_0^t E(t-\tau) e^{-\alpha\tau} \left( \cos \omega_d \tau - \frac{\alpha}{\omega_d} \sin \omega_d \tau \right) d\tau \text{ in the underdamped case } (\omega_0 > \alpha)$$

$$I(t) = \frac{1}{L} \int_0^t E(t-\tau)e^{-\alpha\tau}(1-\alpha\tau) d\tau \text{ in the critically damped case } (\omega_0 = \alpha)$$

$$I(t) = \frac{1}{L} \int_0^t E(t-\tau)e^{-\alpha\tau} \left( \cosh \omega_r \tau - \frac{\alpha}{\omega_r} \sinh \omega_r \tau \right) d\tau \text{ in the overdamped case } (\omega_0 < \alpha)$$

where  $\omega_r = \sqrt{\alpha^2 - \omega_0^2}$ , and cosh and sinh are the usual hyperbolic functions.

### Sinusoidal steady state

Sinusoidal steady state is represented by letting  $s = j\omega$ , where  $j$  is the imaginary unit.

Taking the magnitude of the above equation with this substitution:

$$|Y(s = j\omega)| = \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

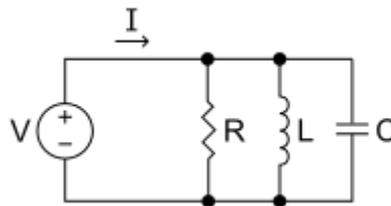
and the current as a function of  $\omega$  can be found from

$$|I(j\omega)| = |Y(j\omega)||V(j\omega)|.$$

There is a peak value of  $|I(j\omega)|$ . The value of  $\omega$  at this peak is, in this particular case, equal to the undamped natural resonance frequency:<sup>[17]</sup>

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

### Parallel RLC circuit



**Figure** RLC parallel circuit

**V** – the voltage of the power source

**I** – the current in the circuit

**R** – the resistance of the resistor

**L** – the inductance of the inductor

**C** – the capacitance of the capacitor

The properties of the parallel RLC circuit can be obtained from the duality relationship of electrical circuits and considering that the parallel RLC is the dual impedance of a series RLC. Considering this it becomes clear that the differential equations describing this circuit are identical to the general form of those describing a series RLC.

For the parallel circuit, the attenuation  $\alpha$  is given by<sup>[18]</sup>

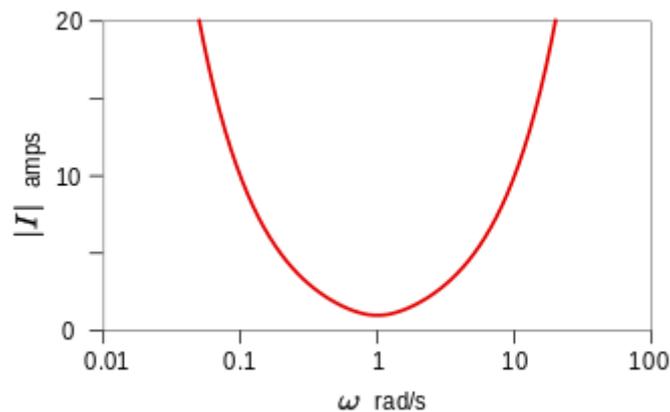
$$\alpha = \frac{1}{2RC}$$

and the damping factor is consequently

$$\zeta = \frac{1}{2R} \sqrt{\frac{L}{C}}$$

Likewise, the other scaled parameters, fractional bandwidth and  $Q$  are also the inverse of each other. This means that a wide band, low  $Q$  circuit in one topology will become a narrow band, high  $Q$  circuit in the other topology when constructed from components with identical values. The  $Q$  and fractional bandwidth of the parallel circuit are given by

$$Q = R \sqrt{\frac{C}{L}} \text{ and } F_b = \frac{1}{R} \sqrt{\frac{L}{C}}$$



**Figure** *Sinusoidal steady-state analysis*

normalised to  $R = 1$  ohm,  $C = 1$  farad,  $L = 1$  henry, and  $V = 1.0$  volt

The complex admittance of this circuit is given by adding up the admittances of the components:

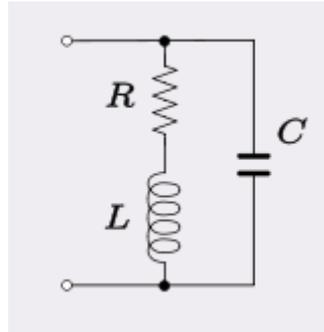
$$\frac{1}{Z} = \frac{1}{Z_L} + \frac{1}{Z_C} + \frac{1}{Z_R} = \frac{1}{j\omega L} + j\omega C + \frac{1}{R}$$

The change from a series arrangement to a parallel arrangement results in the circuit having a peak in impedance at resonance rather than a minimum, so the circuit is an antiresonator.

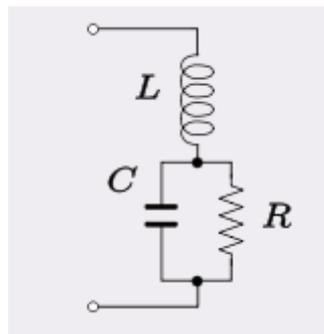
The graph opposite shows that there is a minimum in the frequency response of the current at

the resonance frequency  $\omega_0 = \frac{1}{\sqrt{LC}}$  when the circuit is driven by a constant voltage. On the other hand, if driven by a constant current, there would be a maximum in the voltage which would follow the same curve as the current in the series circuit.

### Other configurations



**Fig.** RLC parallel circuit with resistance in series with the inductor



**Fig.** RLC series circuit with resistance in parallel with the capacitor

A series resistor with the inductor in a parallel LC circuit as shown in figure 7 is a topology commonly encountered where there is a need to take into account the resistance of the coil winding. Parallel LC circuits are frequently used for bandpass filtering and the  $Q$  is largely governed by this resistance. The resonant frequency of this circuit is,<sup>[19]</sup>

$$\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$

This is the resonant frequency of the circuit defined as the frequency at which the admittance has zero imaginary part. The frequency that appears in the generalised form of the characteristic equation (which is the same for this circuit as previously)

$$s^2 + 2\alpha s + \omega_0'^2 = 0$$

is not the same frequency. In this case it is the natural undamped resonant frequency<sup>[20]</sup>

$$\omega'_0 = \sqrt{\frac{1}{LC}}$$

The frequency  $\omega_m$  at which the impedance magnitude is maximum is given by,<sup>[21]</sup>

$$\omega_m = \omega'_0 \sqrt{\frac{-1}{Q_L^2} + \sqrt{1 + \frac{2}{Q_L^2}}}$$

where  $Q_L = \frac{\omega'_0 L}{R}$  is the quality factor of the coil. This can be well approximated by,<sup>[21]</sup>

$$\omega_m \approx \omega'_0 \sqrt{1 - \frac{1}{2Q_L^4}}$$

Furthermore, the exact maximum impedance magnitude is given by,<sup>[21]</sup>

$$|Z|_{max} = RQ_L^2 \sqrt{\frac{1}{2Q_L \sqrt{Q_L^2 + 2} - 2Q_L^2 - 1}}$$

For values of  $Q_L$  greater than unity, this can be well approximated by,<sup>[21]</sup>

$$|Z|_{max} \approx RQ_L^2$$

In the same vein, a resistor in parallel with the capacitor in a series LC circuit can be used to represent a capacitor with a lossy dielectric. This configuration is shown in figure 8. The resonant frequency (frequency at which the impedance has zero imaginary part) in this case is given by,<sup>[22]</sup>

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{1}{(RC)^2}}$$

while the frequency  $\omega_m$  at which the impedance magnitude is maximum is given by

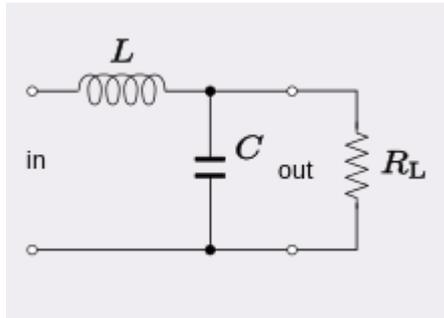
$$\omega_m = \omega'_0 \sqrt{\frac{-1}{Q_C^2} + \sqrt{1 + \frac{2}{Q_C^2}}}$$

where  $Q_C = \omega'_0 RC$

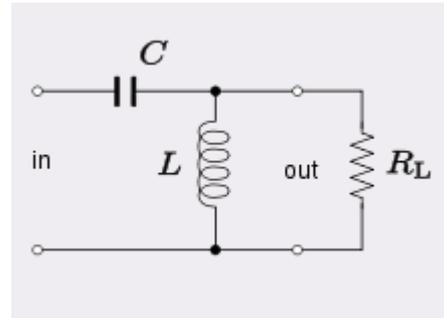
## Applications

### Variable tuned circuits

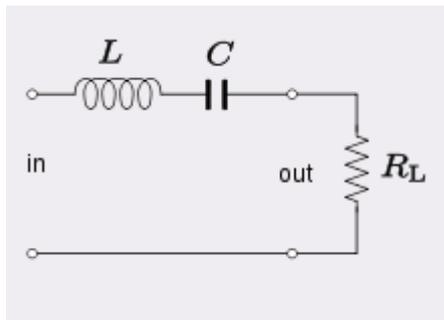
A very frequent use of these circuits is in the tuning circuits of analogue radios. Adjustable tuning is commonly achieved with a parallel plate variable capacitor which allows the value of  $C$  to be changed and tune to stations on different frequencies. For the IF stage in the radio where the tuning is preset in the factory the more usual solution is an adjustable core in the



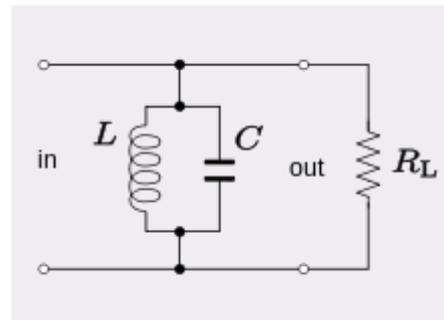
**Fig.** RLC circuit as a low-pass filter



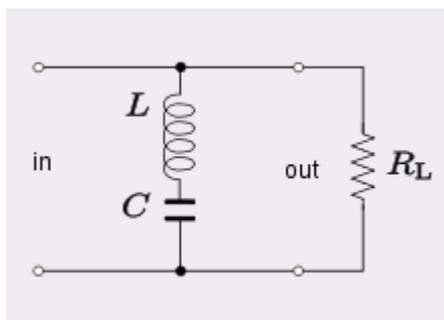
**Fig.** RLC circuit as a high-pass filter



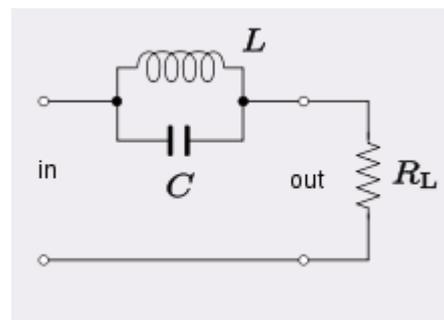
**Fig.** RLC circuit as a series band-pass filter in series with the line



**Fig.** RLC circuit as a parallel band-pass filter in shunt across the line



**Fig.** RLC circuit as a series band-stop filter in shunt across the line



**Fig.** RLC circuit as a parallel band-stop filter in series with the line

inductor to adjust  $L$ . In this design the core (made of a high permeability material that has the effect of increasing inductance) is threaded so that it can

be screwed further in, or screwed further out of the inductor winding as required.

## Foster's reactance theorem:

Network synthesis involves the methods used to determine an electric circuit that satisfy certain specifications. Given an impulse response there are many techniques that can be used to synthesize a circuit with the specified response. Different methods may also be used to synthesize circuits, all of which may be optimal. Hence the solution to a network synthesis problem is never unique.

Many applications today use digital processing in lieu of analog processing and the GHz spectrum is finding increasing use in applications such as wireless communications. However, operation at high frequencies requires analog filtering and processing circuits simply because using digital techniques is neither realistic nor economical. Another advantage that analog devices have over their digital counterparts is their ability to operate with wide instantaneous bandwidths and moderately high dynamic ranges at microwave frequencies.

For a **Foster 1** realisation the component values are given by the partial fraction expansion

$$Z(s) = K_{inf}s + \frac{K_0}{s} + \frac{K_1s}{s^2 + w_1^2} + \frac{K_2s}{s^2 + w_2^2} + \dots + \frac{K_ns}{s^2 + w_n^2}$$

While for **the Foster 2** form the values are given by the alternative partial fraction expansion

$$Y(s) = K'_{inf}s + \frac{K'_0}{s} + \frac{K'_1s}{s^2 + w_1'^2} + \frac{K'_2s}{s^2 + w_2'^2} + \dots + \frac{K'_ns}{s^2 + w_n'^2}$$

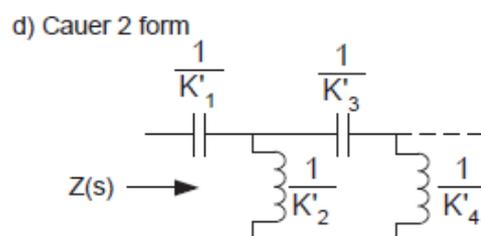
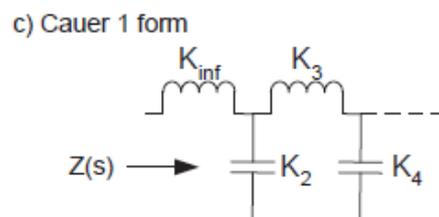
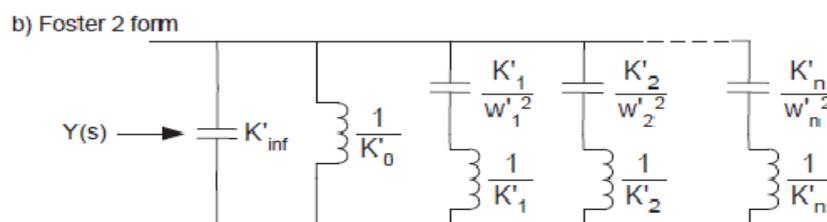
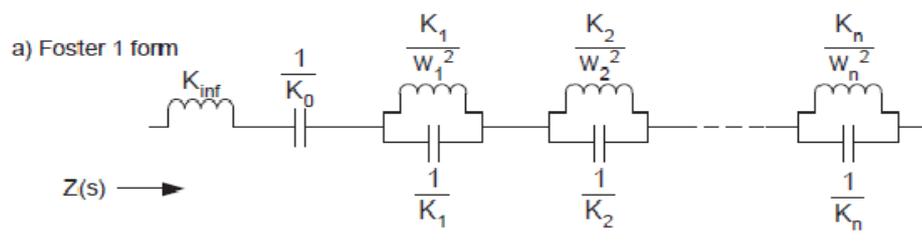
For the **Cauer 1** realization the component values are given by a continued fraction expansion around infinity

$$Z(s) = K_{inf}s + \frac{1}{K_2s + \frac{1}{K_3s + \dots}}$$

The **Cauer 2** values are given by a continued fraction expansion around zero

$$Z(s) = \frac{K'_1}{s} + \frac{1}{\frac{K'_2}{s} + \frac{1}{\frac{K'_3}{s} + \dots}}$$

Foster and Cauer network realisations. These allow simple determination of the Required component values by continued and partial fraction expansions



**Foster's reactance theorem** is an important theorem in the fields of electrical network analysis and synthesis. The theorem states that the reactance of a passive, lossless two-terminal (one-port) network always strictly monotonically increases with frequency. It is easily seen that the reactances of inductors and capacitors individually increase with frequency and from that basis a proof for passive lossless networks generally can be constructed. The proof of the theorem was presented by Ronald Martin Foster in 1924,

although the principle had been published earlier by Foster's colleagues at American Telephone & Telegraph.

The theorem can be extended to admittances and the encompassing concept of immittances. A consequence of Foster's theorem is that poles and zeroes of the reactance must alternate with frequency. Foster used this property to develop two canonical forms for realising these networks. Foster's work was an important starting point for the development of network synthesis.

It is possible to construct non-Foster networks using active components such as amplifiers. These can generate an impedance equivalent to a negative inductance or capacitance. The negative impedance converter is an example of such a circuit.

## MODULE-VII

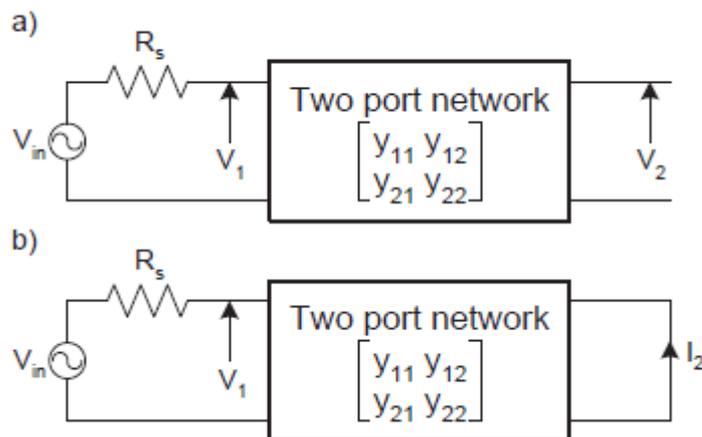
### LC ladder networks :

Analog circuits with passive elements are generally preferred unlike active components, as they do not require an excitation source. Passive LC networks are also more Advantageous as compared to active networks since they have a high tolerance to

Component variances and are simple. Also analog passive circuits can be used as prototypes for designing active networks, interface circuits, transmission lines and other complex networks with discrete components or on chips.

Most importantly, passive LC circuits generally operate in the range of 10<sup>2</sup> to 10<sup>9</sup> Hz . As we will be dealing with high frequency applications (of the order of GHz) in this project, we felt that it was best to use analog passive LC circuits.

$$A(s) = \frac{y_{12}}{y_{11} + 1}$$



$$T(s) = \frac{N(s)}{D_e(s) + D_o(s)}$$

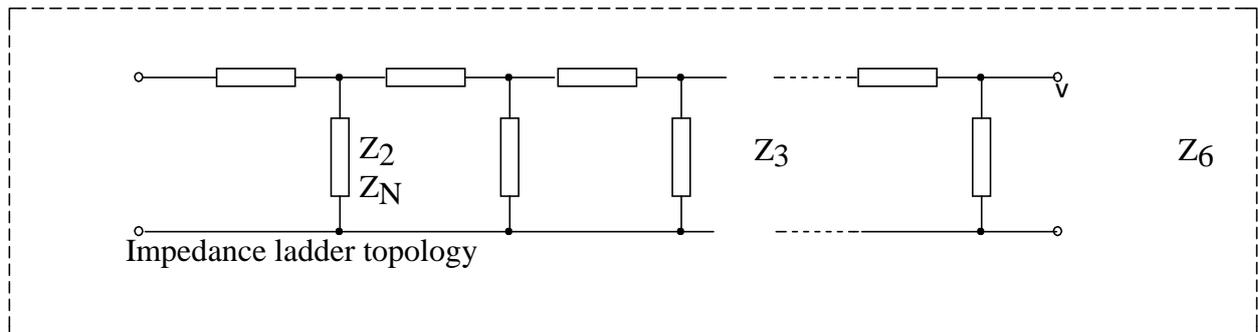
Where  $N(s)$  is an even function and  $D(s)$  is a strictly Hurwitz polynomial. Separating  $D(s)$  into its odd  $D_o(s)$  and even  $D_e(s)$  parts

$$y_{11} = \frac{D_e(s)}{D_o(s)}$$

$$y_{12} = \frac{N(s)}{D_o(s)}$$

## RC circuit applications: RC ladder.

Networks having capacitances and inductances will have both frequency and phase response, both of which may be advantageous to the engineer in circuit applications. The simple cascaded networks are in the form of a “ladder topology”, which can be *RC*, *RL*, or *RLC* constructs.



The elements within the ladder can be a string of simple components, or each impedance can be a more complex option of *RLC* constructs and active components. For this experiment we will confine the analysis of the ladder to simple *RC* cascaded options, and determine frequency and phase characteristics to the ladder, node-by-node and stage-by-stage.